Duality in Linear Programming

4

In the preceding chapter on sensitivity analysis, we saw that the shadow-price interpretation of the optimal simplex multipliers is a very useful concept. First, these shadow prices give us directly the marginal worth of an additional unit of any of the resources. Second, when an activity is ''priced out'' using these shadow prices, the opportunity cost of allocating resources to that activity relative to other activities is determined. Duality in linear programming is essentially a unifying theory that develops the relationships between a given linear program and another related linear program stated in terms of variables with this shadow-price interpretation. The importance of duality is twofold. First, fully understanding the shadow-price interpretation of the optimal simplex multipliers can prove very useful in understanding the implications of a particular linear-programming model. Second, it is often possible to solve the related linear program with the shadow prices as the variables in place of, or in conjunction with, the original linear program, thereby taking advantage of some computational efficiencies. The importance of duality for computational procedures will become more apparent in later chapters on network-flow problems and large-scale systems.

4.1 A PREVIEW OF DUALITY

We can motivate our discussion of duality in linear programming by considering again the simple example given in Chapter 2 involving the firm producing three types of automobile trailers. Recall that the decision variables are:

$$x_1 = \text{number of flat-bed trailers produced per month,}$$
$$x_2 = \text{number of economy trailers produced per month,}$$
$$x_3 = \text{number of luxury trailers produced per month.}$$

The constraining resources of the production operation are the metalworking and woodworking capacities measured in days per month. The linear program to maximize contribution to the firm's overhead (in hundreds of dollars) is:

$$\text{Maximize } z = 6x_1 + 14x_2 + 13x_3,$$

subject to:

$$\begin{aligned} \tfrac{1}{2}x_1 + \ 2x_2 + \ \ x_3 &\leq 24, \\ x_1 + \ 2x_2 + 4x_3 &\leq 60, \\ x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 &\geq 0. \end{aligned} \tag{1}$$

After adding slack variables, the initial tableau is stated in canonical form in Tableau 1.

In Chapter 2, the example was solved in detail by the simplex method, resulting in the final tableau, repeated here as Tableau 2.

Tableau 1

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5
x_4	24	$\frac{1}{2}$	2	1	1	
x_5	60	1	2	4		1
$(-z)$	0	6	14	13		

Tableau 2

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5
x_1	36	1	6		4	-1
x_3	6		-1	1	-1	$\frac{1}{2}$
$(-z)$	-294		-9		-11	$-\frac{1}{2}$

As we saw in Chapter 3, the shadow prices, y_1 for metalworking capacity and y_2 for woodworking capacity, can be determined from the final tableau as the negative of the reduced costs associated with the slack variables x_4 and x_5. Thus these shadow prices are $y_1 = 11$ and $y_2 = \frac{1}{2}$, respectively.

We can interpret the shadow prices in the usual way. One additional day of metalworking capacity is worth \$1100, while one additional day of woodworking capacity is worth only \$50. These values can be viewed as the breakeven rents that the firm could pay per day for additional capacity of each type. If additional capacity could be rented for *less* than its corresponding shadow price, it would be profitable to expand capacity in this way. Hence, in allocating the scarce resources to the production activities, we have determined shadow prices for the resources, which are the values imputed to these resources at the margin.

Let us examine some of the economic properties of the shadow prices associated with the resources. Recall, from Chapter 3, Eq. (11), that the reduced costs are given in terms of the shadow prices as follows:

$$\overline{c}_j = c_j - \sum_{i=1}^{m} a_{ij} y_i \qquad (j = 1, 2, \ldots, n).$$

Since a_{ij} is the amount of resource i used per unit of activity j, and y_i is the imputed value of that resource, the term

$$\sum_{i=1}^{m} a_{ij} y_i$$

is the total value of the resources used per unit of activity j. It is thus the marginal resource cost for using that activity. If we think of the objective coefficients c_j as being marginal revenues, the reduced costs $\overline{c}_j$ are simply net marginal revenues (i.e., marginal revenue minus marginal cost).

For the basic variables x_1 and x_3, the reduced costs are zero,

$$\begin{aligned} \overline{c}_1 &= 6 - 11(\tfrac{1}{2}) - \tfrac{1}{2}(1) = 0, \\ \overline{c}_3 &= 13 - 11(1) - \tfrac{1}{2}(4) = 0. \end{aligned}$$

The values imputed to the resources are such that the net marginal revenue is zero on those activities operated at a positive level. That is, for any production activity at positive level, *marginal revenue must equal marginal cost.*

The situation is much the same for the nonbasic variables x_2, x_4, and x_5, with corresponding reduced costs:

$$\begin{aligned}\overline{c}_2 &= 14 - 11(2) - \tfrac{1}{2}(2) = -9,\\ \overline{c}_4 &= 0 - 11(1) - \tfrac{1}{2}(0) = -11,\\ \overline{c}_5 &= 0 - 11(0) - \tfrac{1}{2}(1) = -\tfrac{1}{2}.\end{aligned}$$

The reduced costs for all nonbasic variables are negative. The interpretation is that, for the values imputed to the scarce resources, *marginal revenue is less than marginal cost* for these activities, so they should not be pursued. In the case of x_2, this simply means that we should not produce *any* economy trailers. The cases of x_4 and x_5 are somewhat different, since slack variables represent unused capacity. Since the marginal revenue of a slack activity is zero, its reduced cost equals *minus its marginal cost*, which is just the shadow price of the corresponding capacity constraint, as we have seen before.

The above conditions interpreted for the reduced costs of the decision variables are the familiar optimality conditions of the simplex method. Economically we can see why they must hold. If marginal revenue exceeds marginal cost for any activity, then the firm would improve its contribution to overhead by *increasing* that activity. If, however, marginal cost exceeds marginal revenue for an activity operated at a positive level, then the firm would increase its contribution by *decreasing* that activity. In either case, a new solution could be found that is an improvement on the current solution. Finally, as we have seen in Chapter 3, those nonbasic variables with zero reduced costs represent possible alternative optimal solutions.

Until now we have used the shadow prices mainly to impute the marginal resource cost associated with each activity. We then selected the best activities for the firm to pursue by comparing the marginal revenue of an activity with its marginal resource cost. In this case, the shadow prices are interpreted as the opportunity costs associated with consuming the firm's resources. If we now value the firm's total resources at these prices, we find their value,

$$v = 11(24) + \tfrac{1}{2}(60) = 294,$$

is exactly equal to the optimal value of the objective function of the firm's decision problem. The implication of this valuation scheme is that the firm's metalworking and woodworking capacities have an imputed worth of \$264 and \$30, respectively. Essentially then, the shadow prices constitute an *internal* pricing system for the firm's resources that:

1. permits the firm to select which activity to pursue by considering only the marginal profitability of its activities; and
2. allocates the contribution of the firm to its resources at the margin.

Suppose that we consider trying to determine directly the shadow prices that satisfy these conditions, without solving the firm's production-decision problem. The shadow prices must satisfy the requirement that marginal revenue be less than or equal to marginal cost for all activities. Further, they must be nonnegative since they are associated with less-than-or-equal-to constraints in a maximization decision problem. Therefore, the unknown shadow prices y_1 on metalworking capacity and y_2 on woodworking capacity must satisfy:

$$\begin{aligned}\tfrac{1}{2}y_1 + y_2 &\geq 6,\\ 2y_1 + 2y_2 &\geq 14,\\ y_1 + 4y_2 &\geq 13,\\ y_1 \geq 0, \qquad y_2 &\geq 0.\end{aligned}$$

These constraints require that the shadow prices be chosen so that the net marginal revenue for each activity is nonpositive. If this were not the case, an improvement in the firm's total contribution could be made by changing the choice of production activities.

Recall that the shadow prices were interpreted as breakeven rents for capacity at the margin. Imagine for the moment that the firm does not own its productive capacity but has to rent it. Now consider any values for the shadow prices, or rental rates, that satisfy the above constraints, say $y_1 = 4$ and $y_2 = 4$. The total worth of the rented capacities evaluated at these shadow prices is $v = 24(4) + 60(4) = 336$, which is greater than the maximum contribution of the firm. Since the imputed value of the firm's resources is derived solely from allocating the firm's contribution to its resources, $v = 336$ is too high a total worth to impute to the firm's resources. The firm clearly could not break even if it had to rent its production capacity at such rates.

If we think of the firm as renting all of its resources, then surely it should try to rent them at least cost. This suggests that we might determine the appropriate values of the shadow prices by minimizing the total rental cost of the resources, subject to the above constraints. That is, solve the following linear program:

$$\text{Minimize } v = 24y_1 + 60y_2,$$

subject to:

$$\begin{aligned} \tfrac{1}{2}y_1 + y_2 &\geq 6, \\ 2y_1 + 2y_2 &\geq 14, \\ y_1 + 4y_2 &\geq 13, \\ y_1 \geq 0, \quad y_2 &\geq 0. \end{aligned} \tag{2}$$

If we solve this linear program by the simplex method, the resulting optimal solution is $y_1 = 11$, $y_2 = \frac{1}{2}$, and $v = 294$. These are exactly the desired values of the shadow prices, and the value of v reflects that the firm's contribution is fully allocated to its resources. Essentially, the linear program (2), in terms of the shadow prices, determines rents for the resources that would allow the firm to break even, in the sense that its total contribution would exactly equal the total rental value of its resources. However, the firm in fact owns its resources, and so the shadow prices are interpreted as the breakeven rates for renting *additional* capacity.

Thus, we have observed that, by solving (2), we can determine the shadow prices of (1) directly. Problem (2) is called the *dual* of Problem (1). Since Problem (2) has a name, it is helpful to have a generic name for the original linear program. Problem (1) has come to be called the *primal.*

In solving any linear program by the simplex method, we also determine the shadow prices associated with the constraints. In solving (2), the shadow prices associated with its constraints are $u_1 = 36$, $u_2 = 0$, and $u_3 = 6$. However, these shadow prices for the constraints of (2) are exactly the optimal values of the decision variables of the firm's allocation problem. Hence, in solving the dual (2) by the simplex method, we apparently have solved the primal (1) as well. As we will see later, this will always be the case since ''the dual of the dual is the primal.'' This is an important result since it implies that the dual may be solved instead of the primal whenever there are computational advantages.

Let us further emphasize the implications of solving these problems by the simplex method. The optimality conditions of the simplex method require that the reduced costs of basic variables be zero. Hence,

$$\begin{aligned} &\text{if } \hat{x}_1 > 0, \quad \text{then } \overline{c}_1 = 6 - \tfrac{1}{2}\hat{y}_1 - \hat{y}_2 = 0; \\ &\text{if } \hat{x}_3 > 0, \quad \text{then } \overline{c}_3 = 13 - \phantom{\tfrac{1}{2}}\hat{y}_1 - 4\hat{y}_2 = 0. \end{aligned}$$

These equations state that, if a decision variable of the primal is positive, then the corresponding constraint in the dual must hold with equality. Further, the optimality conditions require that the nonbasic variables be zero (at least for those variables with negative reduced costs); that is,

$$\text{if } \overline{c}_2 = 14 - 2\hat{y}_1 - 2\hat{y}_2 < 0, \quad \text{then } \hat{x}_2 = 0.$$

These observations are often referred to as *complementary slackness* conditions since, if a variable is positive, its corresponding (complementary) dual constraint holds with equality while, if a dual constraint holds with strict inequality, then the corresponding (complementary) primal variable must be zero.

These results are analogous to what we have seen in Chapter 3. If some shadow price is positive, then the corresponding constraint must hold with equality; that is,

$$\text{if } \hat{y}_1 > 0, \quad \text{then } \tfrac{1}{2}\hat{x}_1 + 2\hat{x}_2 + \hat{x}_3 = 24;$$
$$\text{if } \hat{y}_2 > 0, \quad \text{then } \hat{x}_1 + 2\hat{x}_2 + 4\hat{x}_3 = 60.$$

Further, if a constraint of the primal is not binding, then its corresponding shadow price must be zero. In our simple example there do not happen to be any nonbinding constraints, other than the implicit nonnegativity constraints. However, the reduced costs have the interpretation of shadow prices on the nonnegativity constraints, and we see that the reduced costs of x_1 and x_3 are appropriately zero.

In this chapter we develop these ideas further by presenting the general theory of duality in linear programming.

4.2 DEFINITION OF THE DUAL PROBLEM

The duality principles we have illustrated in the previous sections can be stated formally in general terms. Let the primal problem be:

Primal

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, 2, \ldots, m), \tag{3}$$
$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n).$$

Associated with this primal problem there is a corresponding dual problem given by:

Dual

$$\text{Minimize } v = \sum_{i=1}^{m} b_i y_i,$$

subject to:

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j \qquad (j = 1, 2, \ldots, n), \tag{4}$$
$$y_i \geq 0 \qquad (i = 1, 2, \ldots, m).$$

These primal and dual relationships can be conveniently summarized as in Fig. 4.1.

Without the variables $y_1, y_2, \ldots, y_m$, this tableau is essentially the tableau form utilized in Chapters 2 and 3 for a linear program. The first m rows of the tableau correspond to the constraints of the primal problem, while the last row corresponds to the objective function of the primal problem. If the variables $x_1, x_2, \ldots x_n$, are ignored, the columns of the tableau have a similar interpretation for the dual problem. The first n columns of the tableau correspond to the constraints of the dual problem, while the last column corresponds to the objective function of the dual problem. Note that there is one dual variable for each explicit constraint in the primal, and one primal variable for each explicit constraint in the dual. Moreover, the dual constraints are the familiar optimality condition of ''pricing out'' a column. They state that, at optimality, no activity should appear to be profitable from the standpoint of its reduced cost; that is,

$$\overline{c}_j = c_j - \sum_{i=1}^{m} a_{ij} y_i \leq 0.$$

Dual variables \ Primal variables	$x_1 \geqq 0$	$x_2 \geqq 0$	$x_3 \geqq 0$	$\cdots$	$x_n \geqq 0$	Primal relation	Min v
$y_1 \geqq 0$	a_{11}	a_{12}	a_{13}	$\cdots$	a_{1n}	$\leqq$	b_1
$y_2 \geqq 0$	a_{21}	a_{22}	a_{23}	$\cdots$	a_{2n}	$\leqq$	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
$y_m \geqq 0$	a_{m1}	a_{m2}	a_{m3}	$\cdots$	a_{mn}	$\leqq$	b_m
Dual Relation	$\geqq$	$\geqq$	$\geqq$		$\geqq$		
Max z	c_1	c_2	c_3	$\cdots$	c_n		

Figure 4.1 Primal and dual relationships.

To illustrate some of these relationships, let us consider an example formulated in Chapter 1. Recall the portfolio-selection problem, where the decision variables are the amounts to invest in each security type:

$$\text{Maximize } z = 0.043x_{\text{A}} + 0.027x_{\text{B}} + 0.025x_{\text{C}} + 0.022x_{\text{D}} + 0.045x_{\text{E}},$$

subject to:

$$\begin{array}{llcl}
\textit{Cash} & x_{\text{A}} + x_{\text{B}} + x_{\text{C}} + x_{\text{D}} + x_{\text{E}} & \leq & 10, \\
\textit{Governments} & x_{\text{B}} + x_{\text{C}} + x_{\text{D}} & \geq & 4, \\
\textit{Quality} & 0.6x_{\text{A}} + 0.6x_{\text{B}} - 0.4x_{\text{C}} - 0.4x_{\text{D}} + 3.6x_{\text{E}} & \leq & 0, \\
\textit{Maturity} & 4x_{\text{A}} + 10x_{\text{B}} - x_{\text{C}} - 2x_{\text{D}} - 3x_{\text{E}} & \leq & 0,
\end{array}$$

$$x_{\text{A}} \geq 0, \quad x_{\text{B}} \geq 0, \quad x_{\text{C}} \geq 0, \quad x_{\text{D}} \geq 0, \quad x_{\text{E}} \geq 0.$$

The dual of this problem can be found easily by converting it to the standard primal formulation given in (3). This is accomplished by multiplying the second constraint by -1, thus changing the ''greater than or equal to'' constraint to a ''less than or equal to'' constraint. The resulting primal problem becomes:

$$\text{Maximize } z = 0.043x_{\text{A}} + 0.027x_{\text{B}} + 0.025x_{\text{C}} + 0.022x_{\text{D}} + 0.045x_{\text{E}},$$

subject to:

$$\begin{array}{rcl}
x_{\text{A}} + x_{\text{B}} + x_{\text{C}} + x_{\text{D}} + x_{\text{E}} & \leq & 10, \\
- x_{\text{B}} - x_{\text{C}} - x_{\text{D}} & \leq & -4, \\
0.6x_{\text{A}} + 0.6x_{\text{B}} - 0.4x_{\text{C}} - 0.4x_{\text{D}} + 3.6x_{\text{E}} & \leq & 0, \\
4x_{\text{A}} + 10x_{\text{B}} - x_{\text{C}} - 2x_{\text{D}} - 3x_{\text{E}} & \leq & 0,
\end{array}$$

$$x_{\text{A}} \geq 0, \quad x_{\text{B}} \geq 0, \quad x_{\text{C}} \geq 0, \quad x_{\text{D}} \geq 0, \quad x_{\text{E}} \geq 0.$$

According to expression (4), the corresponding dual problem is:

$$\text{Minimize } v = 10y_1 - 4y_2,$$

subject to:

$$\begin{array}{rcl}
y_1 + 0.6y_3 + 4y_4 & \geq & 0.043, \\
y_1 - y_2 + 0.6y_3 + 10y_4 & \geq & 0.027, \\
y_1 - y_2 - 0.4y_3 - y_4 & \geq & 0.025, \\
y_1 - y_2 - 0.4y_3 - 2y_4 & \geq & 0.022, \\
y_1 + 3.6y_3 - 3y_4 & \geq & 0.045,
\end{array}$$

$$y_1 \geq 0, \qquad y_2 \geq 0, \qquad y_3 \geq 0, \qquad y_4 \geq 0.$$

By applying the simplex method, the optimal solution to both primal and dual problems can be found to be:*

Primal: $x_A = 3.36, \quad x_B = 0, \quad x_C = 0, \quad x_D = 6.48,$
$x_E = 0.16, \quad \text{and } z = 0.294;$

Dual: $y_1 = 0.0294, \quad y_2 = 0, \quad y_3 = 0.00636,$
$y_4 = 0.00244, \quad \text{and } v = 0.294.$

As we have seen before, the optimal values of the objective functions of the primal and dual solutions are equal. Furthermore, an optimal dual variable is nonzero only if its associated constraint in the primal is binding. This should be intuitively clear, since the optimal dual variables are the shadow prices associated with the constraints. These shadow prices can be interpreted as values imputed to the scarce resources (binding constraints), so that the value of these resources equals the value of the primal objective function.

To further develop that the optimal dual variables are the shadow prices discussed in Chapter 3, we note that they satisfy the optimality conditions of the simplex method. In the final tableau of the simplex method, the reduced costs of the basic variables must be zero. As an example, consider basic variable x_A. The reduced cost of x_A in the final tableau can be determined as follows:

$$\begin{aligned}\overline{c}_A &= c_A - \sum_{i=1}^{5} a_{iA} y_i \\ &= 0.043 - 1(0.0294) - 0(0) - 0.6(0.00636) - 4(0.00244) = 0.\end{aligned}$$

For nonbasic variables, the reduced cost in the final tableau must be nonpositive in order to ensure that no improvements can be made. Consider nonbasic variable x_B, whose reduced cost is determined as follows:

$$\begin{aligned}\overline{c}_B &= c_B - \sum_{i=1}^{5} a_{iB} y_i \\ &= 0.027 - 1(0.0294) - 1(0) - 0.6(0.00636) - 10(0.00244) = -0.0306.\end{aligned}$$

The remaining basic and nonbasic variables also satisfy these optimality conditions for the simplex method. Therefore, the optimal dual variables must be the shadow prices associated with an optimal solution.

Since any linear program can be put in the form of (3) by making simple transformations similar to those used in this example, then any linear program must have a dual linear program. In fact, since the dual problem (4) is a linear program, it must also have a dual. For completeness, one would hope that the dual of the dual is the primal (3), which is indeed the case. To show this we need only change the dual (4) into the form of (3) and apply the definition of the dual problem. The dual may be reformulated as a maximization problem with less-than-or-equal-to constraints, as follows:

$$\text{Maximize } v' = \sum_{i=1}^{m} -b_i y_i,$$

subject to:

$$\begin{aligned}\sum_{i=1}^{m} -a_{ij} y_i &\leq -c_j \qquad (j = 1, 2, \ldots, n), \\ y_i &\geq 0 \qquad (i = 1, 2, \ldots, m).\end{aligned} \tag{5}$$

* Excel spreadsheet available at `http://web.mit.edu/15.053/www/Sect4.2_Primal_Dual.xls`

Applying the definition of the dual problem and letting the dual variables be x_j, $j = 1, 2, \ldots, n$, we have

$$\text{Minimize } z' = \sum_{j=1}^{n} -c_j x_j,$$

subject to:

$$\sum_{j=1}^{n} - a_{ij} x_j \geq -b_i \qquad (i = 1, 2, \ldots, m), \tag{6}$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n),$$

which, by multiplying the constraints by minus one and converting the objective function to maximization, is clearly equivalent to the primal problem. Thus, the *dual of the dual is the primal*.

4.3 FINDING THE DUAL IN GENERAL

Very often linear programs are encountered in equality form with nonnegative variables. For example, the canonical form, which is used for computing a solution by the simplex method, is in equality form. It is of interest, then, to find the dual of the equality form:

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j = b_i \qquad (i = 1, 2, \ldots, m), \tag{7}$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n).$$

A problem in equality form can be transformed into inequality form by replacing each equation by two inequalities. Formulation (7) can be rewritten as

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\left.\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\leq b_i \\ \sum_{j=1}^{n} -a_{ij} x_j &\leq -b_i \end{aligned}\right\} \qquad (i = 1, 2, \ldots, m),$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n).$$

The dual of the equality form can then be found by applying the definition of the dual to this problem. Letting y_i^+ and y_i^- $(i = 1, 2, \ldots, m)$ be the dual variables associated with the first m and second m constraints, respectively, from expression (4), we find the dual to be:

$$\text{Minimize } v = \sum_{i=1}^{m} b_i y_i^+ + \sum_{i=1}^{m} -b_i y_i^-,$$

subject to:

$$\sum_{i=1}^{m} a_{ij} y_i^+ + \sum_{i=1}^{m} -a_{ij} y_i^- \geq c_j \qquad (j = 1, 2, \ldots, n),$$

$$y_i^+ \geq 0, \quad y_i^- \geq 0 \qquad (i = 1, 2, \ldots, m).$$

Collecting terms, we have:

$$\text{Minimize } v = \sum_{i=1}^{m} b_i(y_i^+ - y_i^-),$$

subject to:

$$\sum_{i=1}^{m} a_{ij}(y_i^+ - y_i^-) \geq c_j \qquad (j = 1, 2, \ldots, n),$$

$$y_i^+ \geq 0, \quad y_i^- \geq 0 \qquad (i = 1, 2, \ldots, m).$$

Letting $y_i = y_i^+ - y_i^-$, and noting that y_i is unrestricted in sign, gives us the dual of the equality form (7):

$$\text{Minimize } v = \sum_{i=1}^{m} b_i y_i,$$

subject to:

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j \qquad (j = 1, 2, \ldots, n), \tag{8}$$

$$y_i \text{ unrestricted} \qquad (i = 1, 2, \ldots, m).$$

Note that the dual variables associated with equality constraints are unrestricted.

There are a number of relationships between primal and dual, depending upon whether the primal problem is a maximization or a minimization problem and upon the types of constraints and restrictions on the variables. To illustrate some of these relationships, let us consider a general maximization problem as follows:

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\leq b_i \qquad (i = 1, 2, \ldots, m'), \\ \sum_{j=1}^{n} a_{ij} x_j &\geq b_i \qquad (i = m'+1, m'+2, \ldots, m''), \\ \sum_{j=1}^{n} a_{ij} x_j &= b_i \qquad (i = m''+1, m''+2, \ldots, m), \\ x_j &\geq 0 \qquad (j = 1, 2, \ldots, n). \end{aligned} \tag{9}$$

We can change the general primal problem to equality form by adding the appropriate slack and surplus variables, as follows:

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$
\begin{aligned}
\sum_{j=1}^{n} a_{ij}x_j + x_{n+i} \quad &= b_i \qquad (i = 1, 2, \ldots, m'),\\
\sum_{j=1}^{n} a_{ij}x_j \quad - x_{n+i} &= b_i \qquad (i = m'+1, m'+2, \ldots, m''),\\
\sum_{j=1}^{n} a_{ij}x_j \quad &= b_i \qquad (i = m''+1, m''+2, \ldots, m),
\end{aligned}
$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n + m'').$$

Letting y_i, y_i', and y_i'' be dual variables associated respectively with the three sets of equations, the dual of (9) is then

$$\text{Minimize } v = \sum_{i=1}^{m'} b_i y_i + \sum_{i=m'+1}^{m''} b_i y_i' + \sum_{i=m''+1}^{m} b_i y_i'',$$

subject to:

$$
\begin{aligned}
\sum_{i=1}^{m'} a_{ij} y_i + \sum_{i=m'+1}^{m''} a_{ij} y_i' + \sum_{i=m''+1}^{m} a_{ij} y_i'' &\geq c_j \qquad (10)\\
y_i \quad &\geq 0,\\
-y_i' \quad &\geq 0,
\end{aligned}
$$

where y_i'' is unrestricted in sign and the last inequality could be written $y_i' \leq 0$. Thus, if the primal problem is a maximization problem, the dual variables associated with the less-than-or-equal-to constraints are nonnegative, the dual variables associated with the greater-than-or-equal-to constraints are nonpositive, and the dual variables associated with the equality constraints are unrestricted in sign.

These conventions reflect the interpretation of the dual variables as shadow prices of the primal problem. A less-than-or-equal-to constraint, normally representing a scarce resource, has a positive shadow price, since the expansion of that resource generates additional profits. On the other hand, a greater-than-or-equal-to constraint usually represents an external requirement (e.g., demand for a given commodity). If that requirement increases, the problem becomes more constrained; this produces a decrease in the objective function and thus the corresponding constraint has a negative shadow price. Finally, changes in the righthand side of an equality constraint might produce either negative or positive changes in the value of the objective function. This explains the unrestricted nature of the corresponding dual variable.

Let us now investigate the duality relationships when the primal problem is cast in *minimization*, rather than maximization form:

$$\text{Minimize } z = \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\begin{aligned} \sum_{j=1}^{n} a_{ij}x_j &\leq b_i \qquad (i = 1, 2, \ldots, m'), \\ \sum_{j=1}^{n} a_{ij}x_j &\geq b_i \qquad (i = m'+1, m'+2, \ldots, m''), \\ \sum_{j=1}^{n} a_{ij}x_j &= b_i \qquad (i = m''+1, m''+2, \ldots, m), \\ x_j &\geq 0 \qquad (j = 1, 2, \ldots, n). \end{aligned} \tag{11}$$

Since the dual of the dual is the primal, we know that the dual of a minimization problem will be a maximization problem. The dual of (11) can be found by performing transformations similar to those conducted previously. The resulting dual of (11) is the following maximization problem:

$$\text{Maximize } v = \sum_{i=1}^{m'} b_i y_i + \sum_{i=m'+1}^{m''} b_i y'_i + \sum_{i=m''+1}^{m} b_i y''_i,$$

subject to:

$$\begin{aligned} \sum_{i=1}^{m'} a_{ij} y_i + \sum_{i=m'+1}^{m''} a_{ij} y'_i + \sum_{i=m''+1}^{m} a_{ij} y''_i &\geq c_j \\ y_i &\leq 0 \qquad (i = 1, 2, \ldots, m'), \\ y'_i &\geq 0 \qquad (i = m'+1, m'+2, \ldots, m''). \end{aligned} \tag{12}$$

Observe that now the sign of the dual variables associated with the inequality constraints has changed, as might be expected from the shadow-price interpretation. In a cost-minimization problem, increasing the available resources will tend to decrease the total cost, since the constraint has been relaxed. As a result, the dual variable associated with a less-than-or-equal-to constraint in a minimization problem is nonpositive. On the other hand, increasing requirements could only generate a cost increase. Thus, a greater-than-or-equal-to constraint in a minimization problem has an associated nonnegative dual variable.

The primal and dual problems that we have just developed illustrate one further duality correspondence. If (12) is considered as the primal problem and (11) as its dual, then unrestricted variables in the primal are associated with equality constraints in the dual.

We now can summarize the general duality relationships. Basically we note that equality constraints in the primal correspond to unrestricted variables in the dual, while inequality constraints in the primal correspond to restricted variables in the dual, where the sign of the restriction in the dual depends upon the combination of objective-function type and constraint relation in the primal. These various correspondences are summarized in Table 4.1. The table is based on the assumption that the primal is a maximization problem. Since the dual of the dual is the primal, we can interchange the words primal and dual in Table 4.1 and the correspondences will still hold.

4.4 THE FUNDAMENTAL DUALITY PROPERTIES

In the previous sections of this chapter we have illustrated many duality properties for linear programming. In this section we formalize some of the assertions we have already made. The reader not interested in the theory should skip over this section entirely, or merely read the statements of the properties and ignore the proofs. We consider the primal problem in inequality form so that the primal and dual problems are symmetric. Thus,

Table 4.1

Primal (Maximize)	*Dual (Minimize)*
ith constraint $\leq$	ith variable ≥ 0
ith constraint $\geq$	ith variable ≤ 0
ith constraint $=$	ith variable unrestricted
jth variable ≥ 0	jth constraint $\geq$
jth variable ≤ 0	jth constraint $\leq$
jth variable unrestricted	jth constraint $=$

any statement that is made about the primal problem immediately has an analog for the dual problem, and conversely. For convenience, we restate the primal and dual problems.

Primal

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\leq b_i \qquad (i = 1, 2, \ldots, m), \\ x_j &\geq 0 \qquad (j = 1, 2, \ldots, n). \end{aligned} \tag{13}$$

Dual

$$\text{Minimize } v = \sum_{i=1}^{m} b_i y_i,$$

subject to:

$$\begin{aligned} \sum_{i=1}^{m} a_{ij} y_i &\geq c_j \qquad (j = 1, 2, \ldots, n), \\ y_i &\geq 0 \qquad (i = 1, 2, \ldots, m). \end{aligned} \tag{14}$$

The first property is referred to as ''weak duality'' and provides a bound on the optimal value of the objective function of either the primal or the dual. Simply stated, the value of the objective function for any feasible solution to the primal maximization problem is bounded from above by the value of the objective function for any feasible solution to its dual. Similarly, the value of the objective function for its dual is bounded from below by the value of the objective function of the primal. Pictorially, we might represent the situation as follows:

Dual feasible	$\downarrow$	v decreasing
Primal feasible	$\uparrow$	z increasing

The sequence of properties to be developed will lead us to the ''strong duality" property, which states that the optimal values of the primal and dual problems are in fact equal. Further, in developing this result, we show how the solution of one of these problems is readily available from the solution of the other.

Weak Duality Property. If $\overline{x}_j$, $j = 1, 2, \ldots, n$, is a feasible solution to the primal problem and $\overline{y}_i$, $i = 1, 2, \ldots, m$, is a feasible solution to the dual problem, then

$$\sum_{j=1}^{n} c_j \overline{x}_j \leq \sum_{i=1}^{m} b_i \overline{y}_i.$$

The weak duality property follows immediately from the respective feasibility of the two solutions. Primal feasibility implies:

$$\sum_{j=1}^{n} a_{ij} \overline{x}_j \leq b_i \qquad (i = 1, 2, \ldots, m) \quad \text{and} \quad \overline{x}_j \geq 0 \qquad (j = 1, 2, \ldots, n),$$

while dual feasibility implies

$$\sum_{i=1}^{m} a_{ij} \overline{y}_i \geq c_j \qquad (j = 1, 2, \ldots, n) \quad \text{and} \quad \overline{y}_i \geq 0 \qquad (i = 1, 2, \ldots, m).$$

Hence, multiplying the ith primal constraint by $\overline{y}_i$ and adding yields:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \overline{x}_j \overline{y}_i \leq \sum_{i=1}^{m} b_i \overline{y}_i,$$

while multiplying the jth dual constraint by $\overline{x}_j$ and adding yields:

$$\sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} \overline{y}_i \overline{x}_j \geq \sum_{j=1}^{n} c_j \overline{x}_j.$$

Since the lefthand sides of these two inequalities are equal, together they imply the desired result that

$$\sum_{j=1}^{n} c_j \overline{x}_j \leq \sum_{i=1}^{m} b_i \overline{y}_i.$$

There are a number of direct consequences of the weak duality property. If we have feasible solutions to the primal and dual problems such that their respective objective functions are equal, then these solutions are optimal to their respective problems. This result follows immediately from the weak duality property, since a dual feasible solution is an upper bound on the optimal primal solution and this bound is attained by the given feasible primal solution. The argument for the dual problem is analogous. Hence, we have an optimality property of dual linear programs.

Optimality Property. If $\hat{x}_j$, $j = 1, 2, \ldots, n$, is a feasible solution to the primal problem and $\hat{y}_i$, $i = 1, 2, \ldots, m$, is a feasible solution to the dual problem, and, further,

$$\sum_{j=1}^{n} c_j \hat{x}_j = \sum_{i=1}^{m} b_i \hat{y}_i,$$

then $\hat{x}_j$, $j = 1, 2, \ldots, n$, is an optimal solution to the primal problem and $\hat{y}_i$, $i = 1, 2, \ldots, m$, is an optimal solution to the dual problem.

Furthermore, if one problem has an unbounded solution, then the dual of that problem is infeasible. This must be true for the primal since any feasible solution to the dual would provide an upper bound on the primal objective function by the weak duality theorem; this contradicts the fact that the primal problem is unbounded. Again, the argument for the dual problem is analogous. Hence, we have an unboundedness property of dual linear programs.

Unboundedness Property. If the primal (dual) problem has an unbounded solution, then the dual (primal) problem is infeasible.

We are now in a position to give the main result of this section, the ''strong duality'' property. The importance of this property is that it indicates that we may in fact solve the dual problem in place of or in conjunction with the primal problem. The proof of this result depends merely on observing that the shadow prices determined by solving the primal problem by the simplex method give a dual feasible solution, satisfying the optimality property given above.

Strong Duality Property. If the primal (dual) problem has a finite optimal solution, then so does the dual (primal) problem, and these two values are equal. That is, $\hat{z} = \hat{v}$ where

$$\hat{z} = \text{Max} \sum_{j=1}^{n} c_j x_j, \qquad \hat{v} = \text{Min} \sum_{i=1}^{m} b_i y_i,$$

subject to: subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \qquad \sum_{i=1}^{m} a_{ij} y_i \geq c_j,$$

$$x_j \geq 0; \qquad y_i \geq 0.$$

Let us see how to establish this property. We can convert the primal problem to the equivalent equality form by adding slack variables as follows:

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i \qquad (i = 1, 2, \ldots, m),$$

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n+m).$$

Suppose that we have applied the simplex method to the linear program and $\hat{x}_j$, $j = 1, 2, \ldots, n$, is the resulting optimal solution. Let $\hat{y}_i$, $i = 1, 2, \ldots, m$, be the shadow prices associated with the optimal solution. Recall that the shadow prices associated with the original constraints are the multiples of those constraints which, when subtracted from the original form of the objective function, yield the form of the objective function in the final tableau [Section 3.2, expression (11)]. Thus the following condition holds:

$$-z + \sum_{j=1}^{n} \overline{c}_j x_j = -\sum_{i=1}^{m} b_i \hat{y}_i, \tag{15}$$

where, due to the optimality criterion of the simplex method, the reduced costs satisfy:

$$\overline{c}_j = c_j - \sum_{i=1}^{m} a_{ij} \hat{y}_i \leq 0 \qquad (j = 1, 2, \ldots, n), \tag{16}$$

and

$$\overline{c}_j = 0 - \hat{y}_i \leq 0 \qquad (j = n+1, n+2, \ldots, n+m). \tag{17}$$

Conditions (16) and (17) imply that $\hat{y}_i$, for $i = 1, 2, \ldots, m$, constitutes a feasible solution to the dual problem. When x_j is replaced by the optimal value $\hat{x}_j$ in expression (15), the term

$$\sum_{j=1}^{n} \overline{c}_j \hat{x}_j$$

is equal to zero, since $\overline{c}_j = 0$ when $\hat{x}_j$ is basic, and $\hat{x}_j = 0$ when $\hat{x}_j$ is nonbasic. Therefore, the maximum value of z, say $\hat{z}$, is given by:

$$-\hat{z} = -\sum_{i=1}^{m} b_i \hat{y}_i.$$

Moreover, since $\hat{x}_j$, for $j = 1, 2, \ldots, n$, is an optimal solution to the primal problem,

$$\sum_{j=1}^{n} c_j \hat{x}_j = \hat{z} = \sum_{i=1}^{m} b_i \hat{y}_i.$$

This is the *optimality property* for the primal feasible solution $\hat{x}_j$, $j = 1, 2, \ldots, n$, and the dual feasible solution $\hat{y}_i$, $i = 1, 2, \ldots, m$, so they are optimal for their respective problems. (The argument in terms of the dual problem is analogous.)

It should be pointed out that it is *not* true that if the primal problem is infeasible, then the dual problem is unbounded. In this case the dual problem may be either unbounded or infeasible. All four combinations of feasibility and infeasibility for primal and dual problems may take place. An example of each is indicated in Table 4.2.

In example (2) of Table 4.2 it should be clear that x_2 may be increased indefinitely without violating feasibility while at the same time making the objective function arbitrarily large. The constraints of the dual problem for this example are clearly infeasible since $y_1 + y_2 \geq 2$ and $y_1 + y_2 \leq -1$ cannot simultaneously hold. A similar observation is made for example (3), except that the primal problem is now infeasible while the dual variable y_1 may be increased indefinitely. In example (4), the fact that neither primal nor dual problem is feasible can be checked easily by multiplying the first constraint of each by minus one.

4.5 COMPLEMENTARY SLACKNESS

We have remarked that the duality theory developed in the previous section is a unifying theory relating the optimal solution of a linear program to the optimal solution of the dual linear program, which involves the shadow prices of the primal as decision variables. In this section we make this relationship more precise by defining the concept of complementary slackness relating the two problems.

> ***Complementary Slackness Property.*** If, in an optimal solution of a linear program, the value of the dual variable (shadow price) associated with a constraint is nonzero, then that constraint must be satisfied with equality. Further, if a constraint is satisfied with strict inequality, then its corresponding dual variable must be zero.

For the primal linear program posed as a maximization problem with less-than-or-equal-to constraints, this means:

i) if $\hat{y}_i > 0$, then $\sum_{j=1}^{n} a_{ij} \hat{x}_j = b_i$;

ii) if $\sum_{j=1}^{n} a_{ij} \hat{x}_j < b_i$, then $\hat{y}_i = 0$.

Table 4.2

1	*Primal feasible* Maximize $z = 2x_1 + x_2,$ subject to: $x_1 + x_2 \leq 4,$ $x_1 - x_2 \leq 2,$ $x_1 \geq 0, \quad x_2 \geq 0.$	*Dual feasible* Minimize $v = 4y_1 + 2y_2,$ subject to: $y_1 + \ y_2 \geq 2,$ $y_1 - \ y_2 \geq 1,$ $y_1 \geq 0, \qquad y_2 \geq 0.$
2	*Primal feasible and unbounded* Maximize $z = 2x_1 + x_2,$ subject to: $x_1 - x_2 \leq 4,$ $x_1 - x_2 \leq 2,$ $x_1 \geq 0, \quad x_2 \geq 0.$	*Dual infeasible* Minimize $v = 4y_1 + 2y_2,$ subject to: $y_1 + \ y_2 \geq 2,$ $-y_1 - \ y_2 \geq 1,$ $y_1 \geq 0, \qquad y_2 \geq 0.$
3	*Primal infeasible* Maximize $z = 2x_1 + x_2,$ subject to: $-x_1 - x_2 \leq -4,$ $x_1 + x_2 \leq 2,$ $x_1 \geq 0, \quad x_2 \geq 0.$	*Dual feasible and unbounded* Minimize $v = -4y_1 + 2y_2,$ subject to: $-y_1 + \ y_2 \geq 2,$ $-y_1 + \ y_2 \geq 1,$ $y_1 \geq 0, \qquad y_2 \geq 0.$
4	*Primal infeasible* Maximize $z = 2x_1 + x_2,$ subject to: $-x_1 + x_2 \leq -4,$ $x_1 - x_2 \leq 2,$ $x_1 \geq 0, \quad x_2 \geq 0.$	*Dual infeasible* Minimize $v = -4y_1 + 2y_2,$ subject to: $-y_1 + \ y_2 \geq 2,$ $y_1 - \ y_2 \geq 1,$ $y_1 \geq 0, \qquad y_2 \geq 0.$

We can show that the complementary-slackness conditions follow directly from the strong duality property just presented. Recall that, in demonstrating the weak duality property, we used the fact that:

$$\sum_{j=1}^{n} c_j \hat{x}_j \leq \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \hat{x}_j \hat{y}_i \leq \sum_{i=1}^{m} b_i \hat{y}_i \tag{18}$$

for any $\hat{x}_j,\ j = 1, 2, \ldots, n$, and $\hat{y}_i,\ i = 1, 2, \ldots, m$, feasible to the primal and dual problems, respectively. Now, since these solutions are not only feasible but optimal to these problems, equality must hold throughout. Hence, considering the righthand relationship in (18), we have:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} \hat{x}_j \hat{y}_i = \sum_{i=1}^{m} b_i \hat{y}_i,$$

which implies:

$$\sum_{i=1}^{m} \left[\sum_{j=1}^{n} a_{ij} \hat{x}_j - b_i \right] \hat{y}_i = 0.$$

Since the dual variables $\hat{y}_i$ are nonnegative and their coefficients

$$\sum_{j=1}^{n} a_{ij}\hat{x}_j - b_i$$

are nonpositive by primal feasibility, this condition can hold only if each of its terms is equal to zero; that is,

$$\left[\sum_{j=1}^{n} a_{ij}\hat{x}_j - b_i\right]\hat{y}_i = 0 \qquad (i = 1, 2, \ldots, m).$$

These latter conditions are clearly equivalent to (i) and (ii) above.

For the dual linear program posed as a minimization problem with greater-than-or-equal-to constraints, the complementary-slackness conditions are the following:

iii) if $\hat{x}_j > 0$, then $\sum_{i=1}^{m} a_{ij} y_i = c_j$,

iv) if $\sum_{i=1}^{m} a_{ij}\hat{y}_i > c_j$, then $\hat{x}_j = 0$.

These conditions also follow directly from the strong duality property by an argument similar to that given above. By considering the lefthand relationship in (18), we can easily show that

$$\left[\sum_{i=1}^{m} a_{ij} y_i - c_j\right] x_j = 0 \qquad (j = 1, 2, \ldots, n),$$

which is equivalent to (iii) and (iv).

The complementary-slackness conditions of the primal problem have a fundamental economic interpretation. If the shadow price of the ith resource (constraint) is strictly positive in the optimal solution $\hat{y}_i > 0$, then we should require that all of this resource be consumed by the optimal program; that is,

$$\sum_{j=1}^{n} a_{ij}\hat{x}_j = b_i.$$

If, on the other hand, the ith resource is not fully used; that is,

$$\sum_{j=1}^{n} a_{ij}\hat{x}_j < b_i,$$

then its shadow price should be zero, $\hat{y}_i = 0$.

The complementary-slackness conditions of the dual problem are merely the optimality conditions for the simplex method, where the reduced cost $\overline{c}_j$ associated with any variable must be nonpositive and is given by

$$\overline{c}_j = c_j - \sum_{i=1}^{m} a_{ij}\hat{y}_i \leq 0 \qquad (j = 1, 2, \ldots, n).$$

If $\hat{x}_j > 0$, then $\hat{x}_j$ must be a basic variable and its reduced cost is defined to be zero. Thus,

$$c_j = \sum_{i=1}^{m} a_{ij}\hat{y}_i.$$

If, on the other hand,

$$c_j - \sum_{i=1}^{m} a_{ij}\hat{y}_i < 0,$$

then $\hat{x}_j$ must be nonbasic and set equal to zero in the optimal solution; $\hat{x}_j = 0$.

We have shown that the strong duality property implies that the complementary-slackness conditions must hold for both the primal and dual problems. The converse of this also is true. If the complementary-slackness conditions hold for both problems, then the strong duality property holds. To see this, let $\hat{x}_j,\ j = 1, 2, \ldots, n$, and $\hat{y}_i,\ i = 1, 2, \ldots, m$, be feasible solutions to the primal and dual problems, respectively. The complementary-slackness conditions (i) and (ii) for the primal problem imply:

$$\sum_{i=1}^{m}\left[\sum_{j=1}^{n} a_{ij}\hat{x}_j - b_i\right]\hat{y}_i = 0,$$

while the complementary-slackness conditions (iii) and (iv) for the dual problem imply:

$$\sum_{j=1}^{n}\left[\sum_{i=1}^{m} a_{ij}\hat{y}_i - c_j\right]\hat{x}_j = 0.$$

These two equations together imply:

$$\sum_{j=1}^{n} c_j\hat{x}_j = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\hat{x}_j\hat{y}_i = \sum_{i=1}^{m} b_i\hat{y}_i,$$

and hence the values of the primal and dual objective functions are equal. Since these solutions are feasible to the primal and dual problems respectively, the *optimality property* implies that these solutions are optimal to the primal and dual problems. We have, in essence, shown that the complementary-slackness conditions holding for both the primal and dual problems is equivalent to the *strong duality property*. For this reason, the complementary-slackness conditions are often referred to as the *optimality conditions*.

> ***Optimality Conditions.*** If $\hat{x}_j$, $j = 1, 2, \ldots, n$, and $\hat{y}_i$, $i = 1, 2, \ldots, m$, are feasible solutions to the primal and dual problems, respectively, then they are optimal solutions to these problems if, and only if, the complementary-slackness conditions hold for both the primal and the dual problems.

4.6 THE DUAL SIMPLEX METHOD

One of the most important impacts of the general duality theory presented in the previous section has been on computational procedures for linear programming. First, we have established that the dual can be solved in place of the primal whenever there are advantages to doing so. For example, if the number of constraints of a problem is much greater than the number of variables, it is usually wise to solve the dual instead of the primal since the solution time increases much more rapidly with the number of constraints in the problem than with the number of variables. Second, new algorithms have been developed that take advantage of the duality theory in more subtle ways. In this section we present the dual simplex method. We have already seen the essence of the dual simplex method in Section 3.5, on righthand-side ranging, where the variable transitions at the boundaries of each range are essentially computed by the dual simplex method. Further, in Section 3.8, on parametric programming of the righthand side, the dual simplex method is the cornerstone of the computational approach. In this section we formalize the general algorithm.

Recall the canonical form employed in the simplex method:

$$\begin{array}{llllll}
x_1 & & & + \ \overline{a}_{1,m+1}x_{m+1} + \cdots + \overline{a}_{1,n}x_n & = \overline{b}_1, \\
& x_2 & & + \ \overline{a}_{2,m+1}x_{m+1} + \cdots + \overline{a}_{2,n}x_n & = \overline{b}_2, \\
& & \ddots & \qquad \vdots \qquad\qquad\qquad \vdots & \ \vdots \\
& & & x_m + \overline{a}_{m,m+1}x_{m+1} + \cdots + \overline{a}_{m,n}x_n & = \overline{b}_m, \\
(-z) & & & + \quad \overline{c}_{m+1}x_{m+1} + \cdots + \quad \overline{c}_n x_n & = -\overline{z}_0.
\end{array}$$

The conditions for $x_1, x_2, \ldots, x_m$ to constitute an optimal basis for a maximization problem are:

$$\text{i)} \qquad \overline{c}_j \leq 0 \qquad (j = 1, 2, \ldots, n),$$

$$\text{ii)} \qquad \overline{b}_i \geq 0 \qquad (i = 1, 2, \ldots, m).$$

We could refer to condition (i) as primal optimality (or equivalently, dual feasibility) and condition (ii) as primal feasibility. In the primal simplex method, we move from basic feasible solution to adjacent basic feasible solution, increasing (not decreasing) the objective function at each iteration. Termination occurs when the primal optimality conditions are satisfied. Alternatively, we could maintain primal optimality (dual feasibility) by imposing (i) and terminating when the primal feasibility conditions (ii) are satisfied. This latter procedure is referred to as the *dual simplex method* and in fact results from applying the simplex method to the dual problem. In Chapter 3, on sensitivity analysis, we gave a preview of the dual simplex method when we determined the variable to leave the basis at the boundary of a righthand-side range. In fact, the dual simplex method is most useful in applications when a problem has been solved and a subsequent change on the righthand side makes the optimal solution no longer primal feasible, as in the case of parametric programming of the righthand-side values.

The rules of the dual simplex method are identical to those of the primal simplex algorithm, except for the selection of the variable to leave and enter the basis. At each iteration of the dual simplex method, we require that:

$$\overline{c}_j = c_j - \sum_{i=1}^{m} y_i a_{ij} \leq 0;$$

and since $y_i \geq 0$ for $i = 1, 2, \ldots, m$, these variables are a dual feasible solution. Further, at each iteration of the dual simplex method, the most negative $\overline{b}_i$ is chosen to determine the pivot row, corresponding to choosing the most positive $\overline{c}_j$ to determine the pivot column in the primal simplex method.

Prior to giving the formal rules of the dual simplex method, we present a simple example to illustrate the essence of the procedure. Consider the following maximization problem with nonnegative variables given in Tableau 3. This problem is in ‘‘dual canonical form" since the optimality conditions are satisfied, but the basic variables are not yet nonnegative.

Table 4.3

	Basic variables	*Current values*	x_1	x_2	x_3	x_4
	x_3	−1	−1	−1	1	
←	x_4	−2	−2	−3		1
	$(-z)$	0	−3	−1		
				↑		

In the dual simplex algorithm, we are attempting to make all variables nonnegative. The procedure is the opposite of the primal method in that it first selects the variable to drop from the basis and then the new variable to introduce into the basis in its place. The variable to drop is the basic variable associated with the constraint with the most negative righthand-side value; in this case x_4. Next we have to determine the entering variable. We select from only those nonbasic variables that have a negative coefficient in the pivot row, since then, after pivoting, the righthand side becomes positive, thus eliminating the primal infeasibility in the constraint. If all coefficients are positive, then the primal problem is clearly infeasible, because the sum of nonnegative terms can never equal the negative righthand side.

In this instance, both nonbasic variables x_1 and x_2 are candidates. Pivoting will subtract some multiple, say t, of the pivot row containing x_4 from the objective function, to give:

$$(-3 + 2t)x_1 + (-1 + 3t)x_2 - tx_4 - z = 2t.$$

Since we wish to maintain the optimality conditions, we want the coefficient of each variable to remain nonpositive, so that:

$$\begin{aligned} -3 + 2t &\leq 0 \qquad \text{(that is, } t \leq \tfrac{3}{2}\text{)}, \\ -1 + 3t &\leq 0 \qquad \text{(that is, } t \leq \tfrac{1}{3}\text{)}, \\ - t &\leq 0 \qquad \text{(that is, } t \geq 0\text{)}. \end{aligned}$$

Setting $t = \frac{1}{3}$ preserves the optimality conditions and identifies the new basic variable with a zero coefficient in the objective function, as required to maintain the dual canonical form.

These steps can be summarized as follows: the variable to leave the basis is chosen by:

$$\overline{b}_r = \min_i \{\overline{b}_i\} = \text{Min}\,\{-1, -2\} = \overline{b}_2.$$

The variable to enter the basis is determined by the dual ratio test:

$$\frac{\overline{c}_s}{\overline{a}_{rs}} = \min_j \left\{ \frac{\overline{c}_j}{\overline{a}_{rj}} \middle| \overline{a}_{rj} < 0 \right\} = \text{Min} \left\{ \frac{-3}{-2}, \frac{-1}{-3} \right\} = \frac{\overline{c}_2}{\overline{a}_{22}}.$$

Pivoting in x_2 in the second constraint gives the new canonical form in Tableau 4.

Table 4.4

Basic variables	*Current values*	x_1	x_2	x_3	x_4
← x_3	$-\frac{1}{3}$	$-\frac{1}{3}$		1	$-\frac{1}{3}$
x_2	$\frac{2}{3}$	$\frac{2}{3}$	1		$-\frac{1}{3}$
$(-z)$	$\frac{2}{3}$	$-\frac{7}{3}$			$-\frac{1}{3}$ ↑

Clearly, x_3 is the leaving variable since $\overline{b}_1 = -\frac{1}{3}$ is the only negative righthand-side coefficient; and x_4 is the entering variable since $\overline{c}_4/\overline{a}_{14}$ is the minimum dualratio in the first row. After pivoting, the new canonical form is given in Tableau 5

Table 4.5

Basic variables	*Current values*	x_1	x_2	x_3	x_4
x_4	1	1		−3	1
x_2	1	1	1	−1	
$(-z)$	1	−2		−1	

Since $\overline{b}_i \geq 0$ for $i = 1, 2$, and $\overline{c}_j \leq 0$ for $j = 1, 2, \ldots, 4$, we have the optimal solution.

Following is a formal statement of the procedure. The proof of it is analogous to that of the primal simplex method and is omitted.

Dual Simplex Algorithm

STEP (0) The problem is initially in canonical form and all $\overline{c}_j \leq 0$.

STEP (1) If $\overline{b}_i \geq 0,\ i = 1, 2, \ldots, m$, then *stop*, we are optimal. If we continue, then there exists some $\overline{b}_i < 0$.

STEP (2) Choose the row to pivot in (i.e., variable to drop from the basis) by:

$$\overline{b}_r = \operatorname*{Min}_i \left\{ \overline{b}_i | \overline{b}_i < 0 \right\}.$$

If $\overline{a}_{rj} \geq 0,\ j = 1, 2, \ldots, n$, then *stop*; the primal problem is infeasible (dual unbounded). If we continue, then there exists $\overline{a}_{rj} < 0$ for some $j = 1, 2, \ldots, n$.

STEP (3) Choose column s to enter the basis by:

$$\frac{\overline{c}_s}{\overline{a}_{rs}} = \operatorname*{Min}_j \left\{ \frac{\overline{c}_j}{\overline{a}_{rj}} \middle| \overline{a}_{rj} < 0 \right\}.$$

STEP (4) Replace the basic variable in row r with variable s and reestablish the canonical form (i.e., pivot on the coefficient $\overline{a}_{rs}$).

STEP (5) Go to Step (1).

Step (3) is designed to maintain $\overline{c}_j \leq 0$ at each iteration, and Step (2) finds the most promising candidate for an improvement in feasibility.

4.7 PRIMAL-DUAL ALGORITHMS

There are many other variants of the simplex method. Thus far we have discussed only the primal and dual methods. There are obvious generalizations that combine these two methods. Algorithms that perform both primal and dual steps are referred to as primal-dual algorithms and there are a number of such algorithms. We present here one simple algorithm of this form called the *parametric primal-dual*. It is most easily discussed in an example.

$$x_1 \geq 0, \qquad x_2 \geq 0,$$

$$\begin{aligned} x_1 + \ x_2 &\leq \ 6, \\ -x_1 + 2x_2 &\leq -\tfrac{1}{2}, \\ x_1 - 3x_2 &\leq -1, \\ -2x_1 + 3x_2 &= z(\max). \end{aligned}$$

The above example can easily be put in canonical form by addition of slack variables. However, neither primal feasibility nor primal optimality conditions will be satisfied. We will arbitrarily consider the above example as a function of the parameter θ in Tableau 6.

Clearly, if we choose θ large enough, this system of equations satisfies the primal feasibility and primal optimality conditions. The idea of the parametric primal-dual algorithm is to choose θ large initially so that these conditions are satisfied, and attempt to reduce θ to zero through a sequence of pivot operations. If we start with $\theta = 4$ and let θ approach zero, primal optimality is violated when $\theta < 3$. If we were to reduce θ below 3, the objective-function coefficient of x_2 would become positive. Therefore we perform a primal simplex pivot by introducing x_2 into the basis. We determine the variable to leave the basis by the minimum-ratio rule of the primal simplex method:

$$\frac{\overline{b}_r}{\overline{a}_{rs}} = \operatorname*{Min}_i \left\{ \frac{\overline{b}_i}{\overline{a}_{is}} \middle| \overline{a}_{is} > 0 \right\} = \left\{ \frac{6}{1}, \frac{2\frac{1}{2}}{2} \right\} = \frac{\overline{b}_2}{\overline{a}_{22}}.$$

x_4 leaves the basis and the new canonical form is then shown in Tableau 7.

Tableau 6

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
x_3	6	1	1	1		
← x_4	$-\frac{1}{2}+\theta$	-1	2		1	
x_5	$-1+\theta$	1	-3			1
$(-z)$	0	-2	$(3-\theta)$			
			↑			

Tableau 7

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
x_3	$6\frac{1}{4}-\frac{1}{2}\theta$	$\frac{3}{2}$		1	$-\frac{1}{2}$	
x_2	$-\frac{1}{4}+\frac{1}{2}\theta$	$-\frac{1}{2}$	1		$\frac{1}{2}$	
← x_5	$-\frac{7}{4}+\frac{5}{2}\theta$	$-\frac{1}{2}$			$\frac{3}{2}$	1
$(-z)$	$-(3-\theta)(-\frac{1}{4}+\frac{1}{2}\theta)$	$(-\frac{1}{2}-\frac{1}{2}\theta)$			$(-\frac{3}{2}+\frac{1}{2}\theta)$	
		↑				

The optimality conditions are satisfied for $\frac{7}{10} \leq \theta \leq 3$. If we were to reduce θ below $\frac{7}{10}$, the righthand-side value of the third constraint would become negative. Therefore, we perform a dual simplex pivot by dropping x_5 from the basis. We determine the variable to enter the basis by the rules of the dual simplex method:

$$\operatorname*{Min}_{j}\left\{\frac{\overline{c}_j}{\overline{a}_{3j}}\middle|\,\overline{a}_{3j}<0\right\}=\left\{\frac{-\frac{1}{2}+-\frac{1}{2}\theta}{-\frac{1}{2}}\right\}=\frac{\overline{c}_1}{\overline{a}_{31}}.$$

After the pivot is performed the new canonical form is given in Tableau 8.

Tableau 8

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5
x_3	$1+7\theta$			1	4	3
x_2	$\frac{3}{2}-2\theta$		1		-1	-1
x_1	$\frac{7}{2}-5\theta$	1			-3	-2
$(-z)$	$-(3-\theta)\left(-\frac{1}{4}+\frac{1}{2}\theta\right)$ $+(\frac{1}{2}+\frac{1}{2}\theta)(\frac{7}{2}-5\theta)$				$(-3-\theta)$	$(-1-\theta)$

As we continue to decrease θ to zero, the optimality conditions remain satisfied. Thus the optimal final tableau for this example is given by setting θ equal to zero.

Primal-dual algorithms are useful when simultaneous changes of both righthand-side and cost coefficients are imposed on a previous optimal solution, and a new optimal solution is to be recovered. When parametric programming of both the objective-function coefficients and the righthand-side values is performed simultaneously, a variation of this algorithm is used. This type of parametric programming is usually referred to as the *rim problem*. (See Appendix B for further details.)

4.8 MATHEMATICAL ECONOMICS

As we have seen in the two previous sections, duality theory is important for developing computational procedures that take advantage of the relationships between the primal and dual problems. However, there is another less obvious area that also has been heavily influenced by duality, and that is mathematical economics.

In the beginning of this chapter, we gave a preview of duality that interpreted the dual variables as shadow prices imputed to the firm's resources by allocating the profit (contribution) of the firm to its resources at the margin. In a perfectly competitive economy, it is assumed that, if a firm could make profits in excess of the value of its resources, then some other firm would enter the market with a lower price, thus tending to eliminate these excess profits. The duality theory of linear programming has had a significant impact on mathematical economics through the interpretation of the dual as the price-setting mechanism in a perfectly competitive economy. In this section we will briefly sketch this idea.

Suppose that a firm may engage in any n production activities that consume and/or produce m resources in the process. Let $x_j \geq 0$ be the level at which the jth activity is operated, and let c_j be the revenue per unit (minus means cost) generated from engaging in the jth activity. Further, let a_{ij} be the amount of the ith resource consumed (minus means produced) per unit level of operation of the jth activity. Assume that the firm starts with a position of b_i units of the ith resource and may buy or sell this resource at a price $y_i \geq 0$ determined by an external market. Since the firm generates revenues and incurs costs by engaging in production activities and by buying and selling resources, its profit is given by:

$$\sum_{j=1}^{n} c_j x_j + \sum_{i=1}^{m} y_i \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right), \tag{19}$$

where the second term includes revenues from selling excess resources and costs of buying additional resources.

Note that if $b_i > \sum_{j=1}^{n} a_{ij} x_j$, the firm sells $b_i - \sum_{j=1}^{n} a_{ij} x_j$ units of resources i to the marketplace at a price y_i. If, however, $b_i < \sum_{j=1}^{n} a_{ij} x_j$, then the firm buys $\sum_{j=1}^{n} a_{ij} x_j - b_i$ units of resource i from the marketplace at a price y_i.

Now assume that the market mechanism for setting prices is such that it tends to minimize the profits of the firm, since these profits are construed to be at the expense of someone else in the market. That is, given x_j for $j = 1, 2, \ldots, n$, the market reacts to minimize (19). Two consequences immediately follow. First, consuming any resource that needs to be purchased from the marketplace, that is,

$$b_i - \sum_{j=1}^{n} a_{ij} x_j < 0,$$

is clearly uneconomical for the firm, since the market will tend to set the price of the resource arbitrarily high so as to make the firm's profits arbitrarily small (i.e., the firm's losses arbitrarily large). Consequently, our imaginary firm will always choose its production activities such that:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, 2, \ldots, m).$$

Second, if any resource were not completely consumed by the firm in its own production and therefore became available for sale to the market, that is,

$$b_i - \sum_{j=1}^{n} a_{ij} x_j > 0,$$

this ''malevolent market" would set a price of zero for that resource in order to minimize the firm's profit. Therefore, the second term of (19) will be zero and the firm's decision problem of choosing production activities so as to maximize profits simply reduces to:

$$\text{Maximize} \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, 2, \ldots, m), \tag{20}$$
$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n),$$

the usual primal linear program.

Now let us look at the problem from the standpoint of the market. Rearranging (19) as follows:

$$\sum_{j=1}^{n} \left(c_j - \sum_{i=1}^{m} a_{ij} y_i \right) x_j + \sum_{i=1}^{m} b_i y_i, \tag{21}$$

we can more readily see the impact of the firm's decision on the market. Note that the term $\sum_{i=1}^{m} a_{ij} y_i$ is the market opportunity cost for the firm using the resources $a_{1j}, a_{2j}, \ldots, a_{mj}$, in order to engage in the jth activity at unit level. Again, two consequences immediately follow. First, if the market sets the prices so that the revenue from engaging in an activity exceeds the market cost, that is,

$$c_j - \sum_{i=1}^{m} a_{ij} y_i > 0,$$

then the firm would be able to make arbitrarily large profits by engaging in the activity at an arbitrarily high level, a clearly unacceptable situation from the standpoint of the market. The market instead will always choose to set its prices such that:

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j \qquad (j = 1, 2, \ldots, n).$$

Second, if the market sets the price of a resource so that the revenue from engaging in that activity does not exceed the potential revenue from the sale of the resources directly to the market, that is,

$$c_j - \sum_{i=1}^{m} a_{ij} y_i < 0,$$

then the firm will not engage in that activity at all. In this latter case, the opportunity cost associated with engaging in the activity is in excess of the revenue produced by engaging in the activity. Hence, the first term of (21) will always be zero, and the market's ‘‘decision" problem of choosing the prices for the resources so as to minimize the firm's profit reduces to:

$$\text{Minimize} \sum_{i=1}^{m} b_i y_i,$$

subject to:

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j \qquad (j = 1, 2, \ldots, n), \tag{22}$$
$$y_i \geq 0 \qquad (i = 1, 2, \ldots, m).$$

The linear program (22) is the *dual* of (20).

The questions that then naturally arise are ‘‘When do these problems have solutions?" and ‘‘What is the relationship between these solutions?" In arriving at the firm's decision problem and the market's ‘‘decision" problem, we assumed that the firm and the market would interact in such a manner that an equilibrium would be arrived at, satisfying:

$$\begin{aligned} \left(b_i - \sum_{j=1}^{n} a_{ij}\hat{x}_j\right)\hat{y}_i &= 0 \qquad (i = 1, 2, \ldots, m), \\ \left(c_j - \sum_{i=1}^{m} a_{ij}\hat{y}_i\right)\hat{x}_j &= 0 \qquad (j = 1, 2, \ldots, n). \end{aligned} \tag{23}$$

These equations are just the *complementary-slackness conditions* of linear programming. The first condition implies that either the *amount* of resource i that is unused (slack in the ith constraint of the primal) is zero, or the *price* of resource i is zero. This is intuitively appealing since, if a firm has excess of a particular resource, then the market should not be willing to pay anything for the surplus of that resource since the maket wishes to minimize the firm's profit. There may be a nonzero market price on a resource only if the firm is consuming all of that resource that is available. The second condition implies that either the amount of excess profit on the jth activity (slack in the jth constraint of the dual) is zero or the level of activity j is zero. This is also appealing from the standpoint of the perfectly competitive market, which acts to eliminate any excess profits.

If we had an equilibrium satisfying (23), then, by equating (19) and (21) for this equilibrium, we can quickly conclude that the extreme values of the primal and dual problems are equal; that is,

$$\sum_{j=1}^{n} c_j\hat{x}_j = \sum_{i=1}^{m} b_i\hat{y}_i.$$

Observe that this condition has the usual interpretation for a firm operating in a perfectly competitive market. It states that the maximum profit that the firm can make equals the market evaluation of its initial endowment of resources. That is, the firm makes no excess profits. The important step is to answer the question of when such equilibrium solutions exist. As we have seen, if the primal (dual) has a finite optimal solution, then so does the dual (primal), and the optimal values of these objective functions are equal. This result is just the strong duality property of linear programming.

4.9 GAME THEORY

The example of the perfectly competitive economy given in the previous section appears to be a game of some sort between the firm and the malevolent market. The firm chooses its strategy to maximize its profits while the market behaves (''chooses" its strategy) in such a way as to minimize the firm's profits. Duality theory is, in fact, closely related to game theory, and in this section we indicate the basic relationships.

In many contexts, a decision-maker does not operate in isolation, but rather must contend with other decision-makers with conflicting objectives. Consider, for example, advertising in a competitive market, portfolio selection in the presence of other investors, or almost any public-sector problem with its multifaceted implications. Each instance contrasts sharply with the optimization models previously discussed, since they concern a single decision-maker, be it an individual, a company, a government, or, in general, any group acting with a common objective and common constraints.

Game theory is one approach for dealing with these ''multiperson" decision problems. It views the decision-making problem as a *game* in which each decision-maker, or *player*, chooses a *strategy* or an action to be taken. When all players have selected a strategy, each individual player receives a *payoff*. As an example, consider the advertising strategies of a two-firm market. There are two players firm R (row player) and firm C (column player). The alternatives open to each firm are its advertising possibilities; payoffs are market shares resulting from the combined advertising selections of both firms. The payoff table in Tableau 9 summarizes the situation.

Tableau 9 Market Share of Firm R

Firm C alternatives / Firm R alternatives	Advertising campaign 1	Advertising campaign 2	Advertising campaign 3
Advertising campaign 1	30%	40%	60%
Advertising campaign 2	20%	10%	30%

Since we have assumed a two-firm market, firm R and firm C share the market, and firm C receives whatever share of the market R does not. Consequently, firm R would like to maximize the payoff entry from the table and firm B would like to minimize this payoff. Games with this structure are called *two-person, zero-sum games*. They are *zero-sum*, since the gain of one player is the loss of the other player.

To analyze the game we must make some behavioral assumptions as to how the players will act. Let us suppose, in this example, that both players are conservative, in the sense that they wish to assure themselves of their possible payoff level regardless of the strategy adopted by their opponent. It selecting its alternative, firm R chooses a row in the payoff table. The worst that can happen from its viewpoint is for firm C to select the minimum column entry in that row. If firm R selects its first alternative, then it can be assured of securing 30% of the market, but no more, whereas if it selects its second alternative it is assured of securing 10%, but no more. Of course, firm R, wishing to secure as much market share as possible, will select alternative 1, to obtain the maximum of these security levels. Consequently, it selects the alternative giving the maximum of the column minimum, known as a *maximin* strategy. Similarly, firm C's security levels are given by the maximum row entries; if it selects alternative 1, it can be assured of losing only 30% of the market, and no more, and so forth. In this way, firm C is led to a *minimax* strategy of selecting the alternative that minimizes its security levels of maximum row entries (see Tableau 10).

Tableau 10 Market Share of Firm R

Firm C / Firm R	Alternative 1	Alternative 2	Alternative 3	Security level for firm R
Alternative 1	30	40	60	30
Alternative 2	20	10	30	10
Security level for firm C	30	40	60	

maximin = 30 (security levels for firm R); minimax = 30 (security levels for firm C)

For the problem at hand, the maximin and minimax are both 30 and we say that the problem has a *saddlepoint*. We might very well expect both players to select the alternatives that lead to this common value—both selecting alternative 1. Observe that, by the way we arrived at this value, the saddlepoint is an *equilibrium* solution in the sense that neither player will move unilaterally from this point. For instance, if firm R adheres to alternative 1, then firm C cannot improve its position by moving to either alternative 2 or 3 since then firm R's market share increases to either 40% or 60%. Similarly, if firm C adheres to alternative 1, then firm R as well will not be induced to move from its saddlepoint alternative, since its market share drops to 20% if it selects alternative 2.

The situation changes dramatically if we alter a single entry in the payoff table (see Tableau 11).

Now the security levels for the two players do not agree, and moreover, given any choice of decisions by the two firms, one of them can always improve its position by changing its strategy. If, for instance, both firms choose alternative 1, then firm R increases its market share from 30% to 60% by switching to alternative 2. After this switch though, it is attractive for firm C then to switch to its second alternative, so that firm R's market share decreases to 10%. Similar movements will continue to take place as indicated by the arrows in the table, and no single choice of alternatives by the players will be ''stable".

Is there any way for the firms to do better in using *randomized strategies*? That is, instead of choosing a strategy outright, a player selects one according to some preassigned probabilities. For example, suppose

Tableau 11 Market Share of Firm R

Firm R \ Firm C	*Alternative 1*	*Alternative 2*	*Alternative 3*	*Security level for firm R*
Alternative 1	30	40	60	30
Alternative 2	60	10	30	10 (maximin = 30)
Security level for firm C	60	40	60	
	minimax = 40			

that firm C selects among its alternatives with probabilities x_1, x_2, and x_3, respectively. Then the expected market share of firm R is

$$30x_1 + 40x_2 + 60x_3 \quad \text{if firm R selects alternative 1,}$$

or

$$60x_1 + 10x_2 + 30x_3 \quad \text{if firm R selects alternative 2.}$$

Since any gain in market share by firm R is a loss to firm C, firm C wants to make the expected market share of firm R as small as possible, i.e., maximize its own expected market share. Firm C can minimize the maximum expected market share of firm R by solving the following linear program.

Minimize v,

subject to:

$$\begin{aligned} 30x_1 + 40x_2 + 60x_3 - v &\leq 0, \\ 60x_1 + 10x_2 + 30x_3 - v &\leq 0, \\ x_1 + x_2 + x_3 &= 1, \\ x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 &\geq 0. \end{aligned} \tag{24}$$

The first two constraints limit firm R's expected market share to be less than or equal to v for each of firm R's pure strategies. By minimizing v, from C limits the expected market share of firm R as much as possible. The third constraint simply states that the chosen probabilities must sum to one. The solution to this linear program is $x_1 = \frac{1}{2}$, $x_2 = \frac{1}{2}$, $x_3 = 0$, and $v = 35$. By using a randomized strategy (i.e., selecting among the first two alternatives with equal probability), firm C has improved its security level. Its expected market share has increased, since the expected market share of firm R has decreased from 40 percent to 35 percent.

Let us now see how firm R might set probabilities y_1 and y_2 on its alternative selections to achieve its best security level. When firm R weights its alternatives by y_1 and y_2, it has an expected market share of:

$$\begin{aligned} &30y_1 + 60y_2 && \text{if firm C selects alternative 1,} \\ &40y_1 + 10y_2 && \text{if firm C selects alternative 2,} \\ &60y_1 + 30y_2 && \text{if firm C selects alternative 3.} \end{aligned}$$

Firm R wants its market share as large as possible, but takes a conservative approach in maximizing its minimum expected market share from these three expressions. In this case, firm R solves the following linear program:

Maximize w,

subject to:

$$\begin{aligned} 30y_1 + 60y_2 - w &\geq 0, \\ 40y_1 + 10y_2 - w &\geq 0, \\ 60y_1 + 30y_2 - w &\geq 0, \\ y_1 + y_2 &= 1, \end{aligned} \tag{25}$$

$$y_1 \geq 0, \qquad y_2 \geq 0.$$

The first three constraints require that, regardless of the alternative selected by firm C, firm R's market share will be at least w, which is then maximized. The fourth constraint again states that the probabilities must sum to one. In this case, firm R acts optimally by selecting its first alternative with probability $y_1 = \frac{5}{6}$ and its second alternative with probability $y_2 = \frac{1}{6}$, giving an expected market share of 35 percent.

Note that the security levels resulting from each linear program are identical. On closer examination we see that (25) is in fact the dual of (24)! To see the duality correspondence, first convert the inequality constraints in (24) to the standard ($\geq$) for a minimization problem by multiplying by -1. Then the dual constraint derived from "pricing-out" x_1, for example, will read $-30y_1 - 60y_2 + w \leq 0$, which is the first constraint in (25). In this simple example, we have shown that two-person, zero-sum games reduce to primal and dual linear programs. This is true in general, so that the results presented for duality theory in linear programming may be used to draw conclusions about two-person, zero-sum games. Historically, this took place in the reverse, since game theory was first developed by John von Neumann in 1928 and then helped motivate duality theory in linear programming some twenty years later.

General Discussion

The general situation for a two-person, zero-sum game has the same characteristics as our simple example. The payoff table in Tableau 12 and the conservative assumption on the player's behavior lead to the primal and dual linear programs discussed below.

Tableau 12 Payoff to Row Player (−Payoff to Column Player)

Row player alternatives \ Column player alternatives	1	2	3	$\cdots$	n
1	a_{11}	a_{12}	a_{13}	$\cdots$	a_{1n}
2	a_{21}	a_{22}	a_{23}		a_{2n}
3	a_{31}	a_{32}	a_{33}		a_{3n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
m	a_{m1}	a_{m2}	a_{m3}	$\cdots$	a_{mn}

The column player must solve the linear program:

$$\text{Minimize } v,$$

subject to:

$$\begin{aligned} a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n - v &\leq 0 \qquad (i = 1, 2, \ldots, m), \\ x_1 + x_2 + \cdots + x_n &= 1, \\ x_j \geq 0 \qquad (j = 1, 2, \ldots, n); \end{aligned}$$

and the row player the dual linear program:

$$\text{Maximize } w,$$

subject to:

$$\begin{aligned} a_{1j}y_1 + a_{2j}y_2 + \cdots + a_{mn}y_m - w &\geq 0 \qquad (j = 1, 2, \ldots, n), \\ y_1 + y_2 + \cdots + y_m &= 1, \\ y_i \geq 0 \qquad (i = 1, 2, \ldots, m). \end{aligned}$$

The optimal values for $x_1, x_2, \ldots, x_n$, and $y_1, y_2, \ldots, y_m$, from these problems are the probabilities that the players use to select their alternatives. The optimal solutions give $(\min v) = (\max w)$ and, as before, these solutions provide a stable equilibrium, since neither player has any incentive for improved value by unilaterally altering its optimal probabilities.

More formally, if the row player uses probabilities y_i and the column player uses probabilities x_j, then the expected payoff of the game is given by:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} y_i a_{ij} x_j.$$

By complementary slackness, the optimal probabilities $\hat{y}_i$ and $\hat{x}_j$ of the dual linear programs satisfy:

$$\hat{y}_i = 0 \qquad \text{if } \sum_{j=1}^{n} a_{ij} \hat{x}_j < \hat{v}$$

and

$$\hat{y}_i = 0 \qquad \text{only if } \sum_{j=1}^{n} a_{ij} \hat{x}_j = \hat{v}.$$

Consequently, multiplying the ith inequality in the primal by $\hat{y}_i$, and adding gives

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \hat{y}_i a_{ij} \hat{x}_j = \sum_{i=1}^{m} \hat{y}_i \hat{v} = \hat{v}, \tag{26}$$

showing that the primal and dual solutions $\hat{x}_j$ and $\hat{y}_i$ lead to the payoff $\hat{v}$. The last equality uses the fact that the probabilities $\hat{y}_i$ sum to 1.

Next, consider any other probabilities y_i for the row player. Multiplying the ith primal equation

$$\sum_{j=1}^{n} a_{ij} \hat{x}_j \leq \hat{v}$$

by y_i and adding, we find that:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} y_i a_{ij} \hat{x}_j \leq \sum_{i=1}^{m} y_i \hat{v} = \hat{v}. \tag{27}$$

Similarly, multiplying the jth dual constraint

$$\sum_{i=1}^{m} \hat{y}_i a_{ij} \geq \hat{w}$$

by any probabilities x_j and adding gives:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \hat{y}_i a_{ij} x_j \geq \sum_{j=1}^{n} \hat{w} x_j = \hat{w}. \tag{28}$$

Since $\hat{v} = \hat{w}$ by linear programming duality theory, Eqs. (26), (27), and (28) imply that:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} y_i a_{ij} \hat{x}_j \leq \sum_{i=1}^{m} \sum_{j=1}^{n} \hat{y}_i a_{ij} \hat{x}_j \leq \sum_{i=1}^{m} \sum_{j=1}^{n} \hat{y}_i a_{ij} x_j.$$

This expression summarizes the equilibrium condition. By unilaterally altering its selection of probabilities $\hat{y}_i$ to y_i, the row player cannot increase its payoff beyond $\hat{v}$.
Similarly, the column player cannot *reduce* the row player's payoff *below* $\hat{v}$ by unilaterally changing *its* selection of probabilities $\hat{x}_j$ to x_j. Therefore, the probabilities $\hat{y}_i$ and $\hat{x}_j$ acts as an equilibrium, since neither player has an incentive to move from these solutions.

EXERCISES

1. Find the dual associated with each of the following problems:

a) Minimize $z = 3x_1 + 2x_2 - 3x_3 + 4x_4$,

subject to:

$$\begin{aligned} x_1 - 2x_2 + 3x_3 + 4x_4 &\leq 3, \\ x_2 + 3x_3 + 4x_4 &\geq -5, \\ 2x_1 - 3x_2 - 7x_3 - 4x_4 &= 2, \end{aligned}$$

$$x_1 \geq 0, \qquad x_4 \leq 0.$$

b) Maximize $z = 3x_1 + 2x_2$,

subject to:

$$\begin{aligned} x_1 + 3x_2 &\leq 3, \\ 6x_1 - x_2 &= 4, \\ x_1 + 2x_2 &\leq 2, \end{aligned}$$

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

2. Consider the linear-programming problem:

Maximize $z = 2x_1 + x_2 + 3x_3 + x_4$,

subject to:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &\leq 5, \\ 2x_1 - x_2 + 3x_3 &= -4, \\ x_1 - x_3 + x_4 &\geq 1, \end{aligned}$$

$$x_1 \geq 0, \quad x_3 \geq 0, \quad x_2 \text{ and } x_4 \text{ unrestricted.}$$

a) State this problem with equality constraints and nonnegative variables.
b) Write the dual to the given problem and the dual to the transformed problem found in part (a). Show that these two dual problems are equivalent.

3. The initial and final tableaus of a linear-programming problems are as follows:

Initial Tableau

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5	x_6
x_5	6	4	9	7	10	1	
x_6	4	1	1	3	40		1
$(-z)$	0	12	20	18	40		

a) Find the optimal solution for the dual problem.
b) Verify that the values of the shadow prices are the dual feasible, i.e., that they satisfy the following relationship:

$$\overline{c}_j = c_j - \sum_{i=1}^{m} a_{ij}\overline{y}_i \leq 0,$$

where the terms with bars refer to data in the final tableau and the terms without bars refer to data in the initial tableau.

Final Tableau

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5	x_6
x_1	$\frac{4}{3}$	1	$\frac{7}{3}$	$\frac{5}{3}$		$\frac{4}{15}$	$-\frac{1}{15}$
x_4	$\frac{1}{15}$		$-\frac{1}{30}$	$\frac{1}{30}$	1	$-\frac{1}{150}$	$\frac{2}{75}$
$(-z)$	$-\frac{56}{3}$		$-\frac{20}{3}$	$-\frac{10}{3}$		$-\frac{44}{15}$	$-\frac{4}{15}$

c) Verify the complementary-slackness conditions.

4. In the second exercise of Chapter 1, we graphically determined the shadow prices to the following linear program:

$$\text{Maximize } z = 2x_1 + x_2,$$

subject to:

$$\begin{aligned} 12x_1 + x_2 &\leq 6, \\ -3x_1 + x_2 &\leq 7, \\ x_2 &\leq 10, \end{aligned}$$

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

a) Formulate the dual to this linear program.
b) Show that the shadow prices solve the dual problem.

5. Solve the linear program below as follows: First, solve the dual problem graphically. Then use the solution to the dual problem to determine which variables in the primal problem are zero in the optimal primal solution. [*Hint*: Invoke complementary slackness.] Finally, solve for the optimal basic variables in the primal, using the primal equations.

Primal

$$\text{Maximize } -4x_2 + 3x_3 + 2x_4 - 8x_5,$$

subject to:

$$\begin{aligned} 3x_1 + x_2 + 2x_3 + x_4 &= 3, \\ x_1 - x_2 + x_4 - x_5 &\geq 2, \end{aligned}$$

$$x_j \geq 0 \qquad (j = 1, 2, 3, 4, 5).$$

6. A dietician wishes to design a minimum-cost diet to meet minimum daily requirements for calories, protein, carbohydrate, fat, vitamin A and vitamin B dietary needs. Several different foods can be used in the diet, with data as specified in the following table.

	Content and costs per pound consumed						
	Food 1	*Food* 2	....	*Food* j	....	*Food* n	*Daily requirements*
Calories	a_{11}	a_{12}		a_{1j}		a_{1n}	b_1
Protein (grams)	a_{21}	a_{22}		a_{2j}		a_{2n}	b_2
Carbohydrate (grams)	a_{31}	a_{32}		a_{3j}		a_{3n}	b_3
Fat (grams)	a_{41}	a_{42}		a_{4j}		a_{4n}	b_4
Vitamin A (milligrams)	a_{51}	a_{52}		a_{5j}		a_{5n}	b_5
Vitamin B (milligrams)	a_{61}	a_{62}		a_{6j}		a_{6n}	b_6
Costs (dollars)	c_1	c_2		c_j		c_n	

a) Formulate a linear program to determine which foods to include in the minimum cost diet. (More than the minimum daily requirements of any dietary need can be consumed.)

b) State the dual to the diet problem, specifying the units of measurement for each of the dual variables. Interpret the dual problem in terms of a druggist who sets prices on the dietary needs in a manner to sell a dietary pill with b_1, b_2, b_3, b_4, b_5, and b_6 units of the given dietary needs at maximum profit.

7. In order to smooth its production scheduling, a footwear company has decided to use a simple version of a linear cost model for aggregate planning. The model is:

$$\text{Minimize } z = \sum_{i=1}^{N}\sum_{t=1}^{T}(v_i X_{it} + c_i I_{it}) + \sum_{t=1}^{T}(rW_t + oO_t),$$

subject to:

$$X_{it} + I_{i,t-1} - I_{it} = d_{it} \quad \begin{cases} t = 1, 2, \ldots, T \\ i = 1, 2, \ldots, N \end{cases}$$

$$\sum_{i=1}^{N} k_i X_{it} - W_t - O_t = 0 \qquad t = 1, 2, \ldots, T$$

$$0 \leq W_t \leq (rm) \qquad t = 1, 2, \ldots, T$$

$$-pW_t + O_t \leq 0 \qquad t = 1, 2, \ldots, T$$

$$X_{it}, I_{it} \geq 0 \quad \begin{cases} i = 1, 2, \ldots, N \\ t = 1, 2, \ldots, T \end{cases}$$

with parameters:

v_i = Unit production cost for product i in each period,

c_i = Inventory-carrying cost per unit of product i in each period,

r = Cost per man-hour of regular labor,

o = Cost per man-hour of overtime labor,

d_{it} = Demand for product i in period t,

k_i = Man-hours required to produce one unit of product i,

(rm) = Total man-hours of regular labor available in each period,

p = Fraction of labor man-hours available as overtime,

T = Time horizon in periods,

N = Total number of products.

The decision variables are:

X_{it} = Units of product i to be produced in period t,

I_{it} = Units of product i to be left over as inventory at the end of period t,

W_t = Man-hours of regular labor used during period (fixed work force),

O_t = Man-hours of overtime labor used during peirod t.

The company has two major products, boots and shoes, whose production it wants to schedule for the next three periods. It costs \$10 to make a pair of boots and \$5 to make a pair of shoes. The company estimates that it costs \$2 to maintain a pair of boots as inventory through the end of a period and half this amount for shoes. Average wage rates, including benefits, are three dollars an hour with overtime paying double. The company prefers a constant labor force and estimates that regular time will make 2000 man-hours available per period. Workers are willing to increase their work time up to 25% for overtime compensation. The demand for boots and shoes for the three periods is estimated as:

Period	*Boots*	*Shoes*
1	300 pr.	3000 pr.
2	600 pr.	5000 pr.
3	900 pr.	4000 pr.

a) Set up the model using 1 man-hour and $\frac{1}{2}$ man-hour as the effort required to produce a pair of boots and shoes, respectively.
b) Write the dual problem.
c) Define the physical meaning of the dual objective function and the dual constraints.

8. In capital-budgeting problems within the firm, there is a debate as to whether or not the appropriate objective function should be discounted. The formulation of the capital-budgeting problem without discounting is as follows:

$$\text{Maximize } v_N,$$

subject to

Shadow prices

$$\sum_{j=1}^{J}(-c_{ij}x_j) \leq f_i \qquad (i = 0, 1, 2, \ldots, N-1)$$

$$\sum_{j=1}^{J}(-c_{Nj}x_j) + v_N \leq f_N, \qquad y_N$$

$$0 \leq x_j \leq u_j \qquad (j = 1, 2, \ldots, J),$$

where c_{ij} is the cash outflow ($c_{ij} < 0$) or inflow ($c_{ij} > 0$) in period i for project j; the righthand-side constant f_i is the net exogenous funds made available to ($f_i > 0$) or withdrawn from ($f_i < 0$) a division of the firm in period i; the decision variable x_j is the level of investment in project j; u_j is an upper bound on the level of investment in project j; and v_N is a variable measuring the value of the holdings of the division at the end of the planning horizon.

If the undiscounted problem is solved by the bounded variable simplex method, the optimal solution is x_j^* for $j = 1, 2, \ldots, J$ with associated shadow prices (dual variables) y_i^* for $i = 0, 1, 2, \ldots, N$.

a) Show that $y_N^* = 1$.
b) The discount factor for time i is defined to be the present value (time $i = 0$) of one dollar received at time i. Show that the discount factor for time i, ρ_i^* is given by:

$$\rho_i^* = \frac{y_i^*}{y_0^*} \qquad (i = 0, 1, 2, \ldots, N).$$

(Assume that the initial budget constraint is binding so that $y_0^* > 0$.)
c) Why should $\rho_0^* = 1$?

9. The *discounted* formulation of the capital-budgeting problem described in the previous exercise can be stated as:

$$\text{Maximize } \sum_{j=1}^{J} \left(\sum_{i=0}^{N} \rho_i c_{ij} \right) x_j,$$

subject to

$$\sum_{j=1}^{J} (-c_{ij} x_j) \leq f_i \qquad (i = 0, 1, 2, \ldots, N), \qquad \lambda_i \quad \text{(Shadow prices)}$$

$$0 \leq x_j \leq u_j \qquad (j = 1, 2, \ldots, J),$$

where the objective-function coefficient

$$\sum_{i=0}^{N} \rho_i c_{ij}$$

represents the discounted present value of the cash flows from investing at unit level in project j, and λ_i for $i = 1, 2, \ldots, N$ are the shadow prices associated with the funds-flow constraints.

Suppose that we wish to solve the above discounted formulation of the capital-budgeting problem, using the discount factors determined by the optimal solution of the previous exercise, that is, setting:

$$\rho_i = \rho_i^* = \frac{y_i^*}{y_0^*} \qquad (i = 0, 1, 2, \ldots, N).$$

Show that the optimal solution x_j^* for $j = 1, 2, \ldots, J$, determined from the undiscounted case in the previous exercise, is also optimal to the above discounted formulation, assuming:

$$\rho_i = \rho_i^* \qquad (i = 0, 1, 2, \ldots, N).$$

[*Hint*: Write the optimality conditions for the discounted problem, using shadow-price values $\lambda_i^* = 0$ for $i = 1, 2, \ldots, N$. Does x_j^* ($j = 1, 2, \ldots, J$) satisfy these conditions? Do the shadow prices on the upper bounding constraints for the discounted model differ from those of the undiscounted model?]

10. An alternative formulation of the undiscounted problem developed in Exercise 8 is to maximize the total *earnings* on the projects rather than the horizon value. Earnings of a project are defined to be the net cash flow from a project over its lifetime. The alternative formulation is then:

$$\text{Maximize } \sum_{j=1}^{J}\left(\sum_{i=0}^{N} c_{ij}\right) x_j,$$

subject to

		Shadow prices
$\sum_{j=1}^{J}(-c_{ij}x_j) \leq f_i$	$(i = 0, 1, 2, \ldots, N),$	y'_i
$0 \leq x_j \leq u_j$	$(j = 1, 2, \ldots, J).$	

Let x_j^* for $j = 1, 2, \ldots, J$ solve this earnings formulation, and suppose that the funds constraints are binding at all times $i = 0, 1, \ldots, N$ for this solution.

a) Show that x_j^* for $j = 1, 2, \ldots, J$ also solves the horizon formulation given in Exercise 8.
b) Denote the optimal shadow prices of the funds constraints for the earnings formulation as y'_i for $i = 0, 1, 2, \ldots, N$. Show that $y_i = 1 + y'_i$ for $i = 0, 1, 2, \ldots, N$ are optimal shadow prices for the funds constraints of the horizon formulation.

11. Suppose that we now consider a variation of the horizon model given in Exercise 8, that explicitly includes one-period borrowing b_i at rate r_b and one-period lending ℓ_i at rate r_ℓ. The formulation is as follows:

$$\text{Maximize } v_N,$$

subject to

		Shadow prices
$\sum_{j=1}^{J}(-c_{0j}x_j) + \ell_0 - b_0 \leq f_0,$		y_0
$\sum_{j=1}^{J}(-c_{ij}x_j) - (1+r_\ell)\ell_{i-1} + \ell_i + (1+r_b)b_{i-1} - b_i \leq f_i$	$(i = 1, 2, \ldots, N-1),$	y_i
$\sum_{j=1}^{J}(-c_{Nj}x_j) - (1+r_\ell)\ell_{N-1} + (1-r_b)b_{N-1} + v_N \leq f_N,$		y_N
$b_i \leq B_i$	$(i = 0, 1, \ldots, N-1),$	w_i
$0 \leq x_j \leq u_j$	$(j = 1, 2, \ldots, J),$	
$b_i \geq 0, \quad \ell_i \geq 0$	$(i = 0, 1, \ldots, N-1).$	

a) Suppose that the optimal solution includes lending in every time period; that is, $\ell_i^* > 0$ for $i = 0, 1, \ldots, N-1$. Show that the present value of a dollar in period i is

$$\rho_i = \left(\frac{1}{1+r_\ell}\right)^i.$$

b) Suppose that there is no upper bound on borrowing; that is, $B_i = +\infty$ for $i = 0, 1, \ldots, N-1$, and that the borrowing rate equals the lending rate, $r = r_b = r_\ell$. Show that

$$\frac{y_{i-1}}{y_i} = 1 + r,$$

and that the present value of a dollar in period i is

$$\rho_i = \left(\frac{1}{1+r}\right)^i.$$

c) Let w_i be the shadow prices on the upper-bound constraints $b_i \leq B_i$, and let $r = r_b = r_\ell$. Show that the shadow prices satisfy:

$$1 + r \leq \frac{y_{i-1}}{y_i} \leq 1 + r + w_{i-1}.$$

d) Assume that the borrowing rate is greater than the lending rate, $r_b > r_\ell$; show that the firm will not borrow and lend in the same period if it uses this linear-programming model for its capital budgeting decisions.

12. As an example of the present-value analysis given in Exercise 8, consider four projects with cash flows and upper bounds as follows:

Project	*End of year 0*	*End of year 1*	*End of year 2*	*Upper bound*
A	−1.00	0.60	0.60	∞
B	−1.00	1.10	0	500
C	0	−1.00	1.25	∞
D	−1.00	0	1.30	∞

A negative entry in this table corresponds to a cash outflow and a positive entry to a cash inflow.

The horizon-value model is formulated below, and the optimal solution and associated shadow prices are given:

x_A	x_B	x_C	x_D	v	*Relation*	RHS	*Shadow prices*
1	1		1		$\leq$	1000	1.35
−0.60	−1.1	1			$\leq$	0	1.25
−0.60		−1.25	−1.30	1	$\leq$	0	1.00
	1				$\leq$	500	0.025
				1	=	z(max)	
500	500	850	0	−1362.5			

Solution

a) Explain exactly how *one* additional dollar at the end of year 2 would be invested to return the shadow price 1.25. Similarly, explain how to use the projects in the portfolio to achieve a return of 1.35 for an additional dollar at the end of year 0.
b) Determine the discount factors for each year, from the shadow prices on the constraints.
c) Find the discounted present value $\sum_{i=0}^{N} \rho_i c_{ij}$ of each of the four projects. How does the sign of the discounted present value of a project compare with the sign of the reduced cost for that project? What is the relationship between the two? Can discounting interpreted in this way be used for decision-making?
d) Consider the two additional projects with cash flows and upper bounds as follows:

Project	*End of year 0*	*End of year 1*	*End of year 2*	*Upper bound*
E	−1.00	0.75	0.55	∞
F	−1.00	0.30	1.10	∞

What is the discounted present value of each project? Both appear promising. Suppose that fun ds are transferred from the current portfolio into project E, will project F still appear promising? Why? Do the discount factors change?

13. In the exercises following Chapter 1, we formulated the absolute-value regression problem:

$$\text{Minimize } \sum_{i=1}^{m} |y_i - x_{i1}\beta_1 - x_{i2}\beta_2 - \cdots - x_{in}\beta_n|$$

as the linear program:

$$\text{Minimize } \sum_{i=1}^{m} (P_i + N_i),$$

subject to:

$$\begin{aligned} x_{i1}\beta_1 + x_{i2}\beta_2 + \cdots + x_{in}\beta_n + P_i - N_i &= y_i s \quad \text{for } i = 1, 2, \ldots, m, \\ P_i \geq 0, \quad N_i &\geq 0 \quad \text{for } i = 1, 2, \ldots, m. \end{aligned}$$

In this formulation, the y_i are measurements of the dependent variable (e.g., income), which is assumed to be explained by independent variables (e.g., level of education, parents' income, and so forth), which are measured as $x_{i1}, x_{i2}, \ldots, x_{in}$. A linear model assumes that y depends linearly upon the β's, as:

$$\hat{y}_i = x_{i1}\beta_1 + x_{i2}\beta_2 + \cdots + x_{in}\beta_n. \tag{29}$$

Given any choice of the parameters $\beta_1, \beta_2, \ldots, \beta_n$, $\hat{y}_i$ is an estimate of y_i. The above formulation aims to minimize the deviations of the estimates of $\hat{y}_i$ from y_i as measured by the sum of absolute values of the deviations. The variables in the linear-programming model are the parameters $\beta_1, \beta_2, \ldots, \beta_n$ as well as the P_i and N_i. The quantities $y_i, x_{i1}, x_{i2}, \ldots, x_{in}$ are known data found by measuring several values for the dependent and independent variables for the linear model (29).

In practice, the number of observations m frequently is much larger than the number of parameters n. Show how we can take advantage of this property by formulating the dual to the above linear program in the dual variables $u_1, u_2, \ldots, u_m$. How can the special structure of the dual problem be exploited computationally?

14. The following tableau is in canonical form for maximizing z, except that one righthand-side value is negative.

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5	x_6
x_1	14	1			2	3	−2
x_2	6		1		1	−2	−2
x_3	−10			1	−1	−2	0
$(-z)$	−40				−1	−4	−2

However, the reduced costs of the nonbasic variables all satisfy the primal optimality conditions. Find the optimal solution to this problem, using the dual simplex algorithm to find a feasible canonical form while maintaining the primal optimality conditions.

15. In Chapter 2 we solved a two-constraint linear-programming version of a trailer-production problem:

$$\text{Maximize } 6x_1 + 14x_2 + 13x_3,$$

subject to:

$$\begin{aligned} \tfrac{1}{2}x_1 + 2x_2 + 4x_3 &\leq 24 \quad \text{(Metalworking capacity)}, \\ x_1 + 2x_2 + 4x_3 &\leq 60 \quad \text{(Woodworking capacity)}, \end{aligned}$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0,$$

obtaining an optimal tableau:

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5
x_1	36	1	6		4	-1
x_3	6		-1	1	-1	$\frac{1}{2}$
$(-z)$	-294		-9		-11	$-\frac{1}{2}$

Suppose that, in formulating this problem, we ignored a constraint limiting the time available in the shop for inspecting the trailers.

a) If the solution $x_1 = 36$, $x_2 = 0$, and $x_3 = 6$ to the original problem satisfies the inspection constraint, is it necessarily optimal for the problem when we impose the inspection constraint?
b) Suppose that the inspection constraint is

$$x_1 + x_2 + x_3 + x_6 = 30,$$

where x_6 is a nonnegative slack variable. Add this constraint to the optimal tableau with x_6 as its basic variable and pivot to eliminate the basic variables x_1 and x_3 from this constraint. Is the tableau now in dual canonical form?
c) Use the dual simplex method to find the optimal solution to the trailer-production problem with the inspection constraint given in part (b).
d) Can the ideas used in this example be applied to solve a linear program whenever a new constraint is added after the problem has been solved?

16. Apply the dual simplex method to the following tableau for maximizing z with nonnegative decision variables $x_1, x_2, \ldots, x_5$.

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5
x_1	-3	1		-1	2	-2
x_2	7		1	3	-4	8
$(-z)$	-5			-1	-5	-6

Is the problem feasible? How can you tell?

17. Consider the linear program:

$$\text{Minimize } z = 2x_1 + x_2,$$

subject to:

$$-4x_1 + 3x_2 - x_3 \geq 16,$$
$$x_1 + 6x_2 + 3x_3 \geq 12,$$

$$x_i \geq 0 \quad \text{for } i = 1, 2, 3.$$

a) Write the associated dual problem.
b) Solve the primal problem, using the dual simplex algorithm.
c) Utilizing the final tableau from part (b), find optimal values for the dual variables y_1 and y_2. What is the corresponding value of the dual objective function?

18. For what values of the parameter θ is the following tableau in canonical form for maximizing the objective value z?

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5
x_1	$-2+\theta$	1		-1	2	-3
x_2	1		1	1	0	1
$(-z)$	-20			$3-\theta$	$4-\theta$	-6

Starting with this tableau, use the parametric primal-dual algorithm to solve the linear program at $\theta = 0$.

19. After solving the linear program:

$$\text{Maximize } z = 5x_1 + 7x_2 + 2x_3,$$

subject to:

$$\begin{aligned} 2x_1 + 3x_2 + x_3 + x_4 \qquad &= 5, \\ \tfrac{1}{2}x_1 + x_2 \qquad\qquad + x_5 &= 1, \end{aligned}$$

$$x_j \geq 0 \quad (j = 1, 2, \ldots, 5),$$

and obtaining the optimal canonical form

Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5
x_3	1		-1	1	1	-4
x_1	2	1	2		0	2
$(-z)$	-12		-1		-2	-2

we discover that the problem was formulated improperly. The objective coefficient for x_2 should have been 11 (not 7) and the righthand side of the first constraint should have been 2 (not 5).

a) How do these modifications in the problem formulation alter the data in the tableau above?
b) How can we use the parametric primal-dual algorithm to solve the linear program after these data changes, starting with x_1 and x_3 as basic variables?
c) Find an optimal solution to the problem after the data changes are made, using the parametric primal-dual algorithm.

20. Rock, Paper, and Scissors is a game in which two players simultaneously reveal no fingers (rock), one finger (paper), or two fingers (scissors). The payoff to player 1 for the game is governed by the following table:

Player 1 \ *Player 2*	*Rock*	*Paper*	*Scissors*
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

Payoff to player 1
(or minus payoff to player 2)

Note that the game is void if both players select the same alternative: otherwise, rock breaks scissors and wins, scissors cut paper and wins, and paper covers rock and wins.

Use the linear-programming formulation of this game and linear-programming duality theory, to show that both players' optimal strategy is to choose each alternative with probability $\frac{1}{3}$.

21. Solve for an optimal strategy for both player 1 and player 2 in a zero-sum two-person game with the following payoff table:

Is the optimal strategy of each player unique?

Player 1 \ Player 2	Alternative 1	Alternative 2	Alternative 3
Alternative 1	1	4	3
Alternative 2	5	2	3

Payoff to player 1
(or minus payoff to player 2)

22. In a game of tic-tac-toe, the first player has three different choices at the first move: the center square, a corner square, or a side square. The second player then has different alternatives depending upon the move of the first player. For instance, if the first player chooses a corner square, the second player can select the center square, the opposite corner, an adjacent corner, an opposite side square, or an adjacent side square. We can picture the possible outcomes for the game in a decision tree as follows:

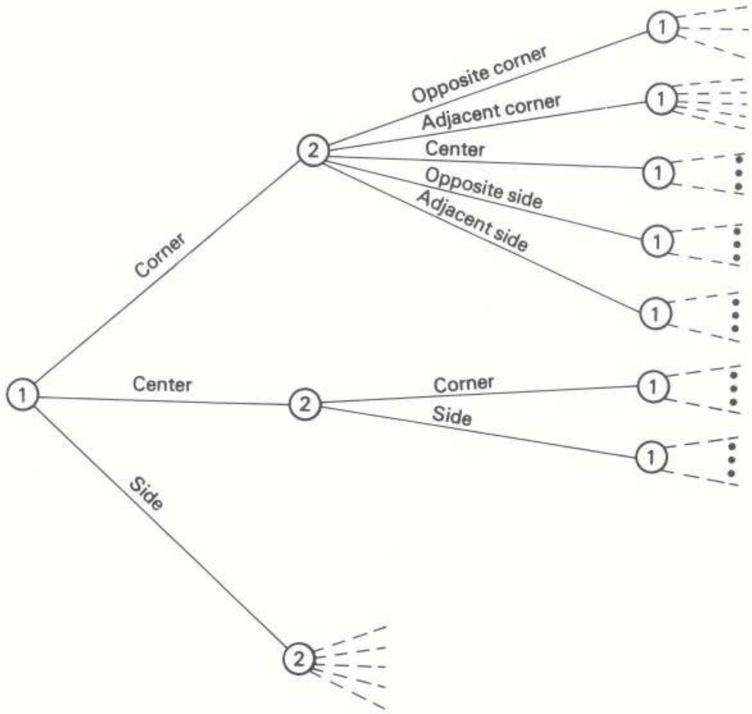

Figure E4.1

Nodes ① correspond to decision points for the first player; nodes ② are decision points for the second player. The tree is constructed in this manner by considering the options available to each player at a given turn, until either one player has won the game (i.e., has selected three adjacent horizontal or vertical squares or the squares along one of the diagonals), or all nine squares have been selected. In the first case, the winning player receives 1 point; in the second case the game is a draw and neither player receives any points.

The decision-tree formulation of a game like that illustrated here is called an *extensive form* formulation. Show how the game can be recast in the linear-programming form (called a *normal form*) discussed in the text. Do not attempt to enumerate every strategy for both players. Just indicate how to construct the payoff tables.

[*Hint*: Can you extract a strategy for player 1 by considering his options at each decision point? Does his strategy depend upon how player 2 reacts to player 1's selections?]

The remaining exercises extend the theory developed in this chapter, providing new results and relating some of the duality concepts to other ideas that arise in mathematical programming.

23. Consider a linear program with bounded variables:

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\begin{array}{lll} & & \underline{\text{Dual variables}} \\ \displaystyle\sum_{j=1}^{n} a_{ij} x_j = b_i & (i = 1, 2, \ldots, m), & y_i \\ x_j \leq u_j & (j = 1, 2, \ldots, n), & w_j \\ x_j \geq 0 & (j = 1, 2, \ldots, n), & \end{array}$$

where the upper bounds u_j are positive constants. Let y_i for $i = 1, 2, \ldots, m$ and w_j for $j = 1, 2, \ldots, m$ be variables in the dual problem.

a) Formulate the dual to this bounded-variable problem.
b) State the optimality conditions (that is, primal feasibility, dual feasibility, and complementary slackness) for the problem.
c) Let $\overline{c}_j = c_j - \sum_{i=1}^{m} y_i a_{ij}$ denote the reduced costs for variable x_j determined by pricing out the a_{ij} constraints and not the upper-bounding constraints. Show that the optimality conditions are equivalent to the bounded-variable optimality conditions

$$\begin{aligned} \overline{c}_j &\leq 0 \quad \text{if } x_j = 0, \\ \overline{c}_j &\geq 0 \quad \text{if } x_j = u_j, \\ \overline{c}_j &= 0 \quad \text{if } 0 < x_j < u_j, \end{aligned}$$

given in Section 2.6 of the text, for any feasible solution x_j $(j = 1, 2, \ldots, n)$ of the primal problem.

24. Let x_j^* for $j = 1, 2, \ldots, n$ be an optimal solution to the linear program with

$$\text{Maximize } \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\begin{array}{lll} & & \underline{\text{Shadow prices}} \\ \displaystyle\sum_{j=1}^{n} a_{ij} x_j \leq b_i & (i = 1, 2, \ldots, m), & y_i \\ \displaystyle\sum_{j=1}^{n} e_{ij} x_j \leq d_i & (i = 1, 2, \ldots, q), & w_i \\ x_j \geq 0 & (j = 1, 2, \ldots, n), & \end{array}$$

two groups of constraints, the a_{ij} constraints and the e_{ij} constraints. Let y_i for $i = 1, 2, \ldots, m$ and w_i for $i = 1, 2, \ldots, q$ denote the shadow prices (optimal dual variables) for the constraints.

a) Show that x_j^* for $j = 1, 2, \ldots, n$ also solves the linear program:

$$\text{Maximize } \sum_{j=1}^{n} \left[c_j - \sum_{i=1}^{m} y_i a_{ij} \right] x_j,$$

subject to:

$$\begin{aligned} \sum_{j=1}^{n} e_{ij} x_j &\leq d_i \quad (i = 1, 2, \ldots, q), \\ x_j &\geq 0 \quad (j = 1, 2, \ldots, n), \end{aligned}$$

in which the a_{ij} constraints have been incorporated into the objective function as in the method of Lagrange multipliers. [*Hint*: Apply the optimality conditions developed in Section 4.5 to the original problem and this ''Lagrangian problem.'']

b) Illustrate the property from part (a) with the optimal solution $x_1^* = 2$, $x_2^* = 1$, and shadow prices $y_1 = \frac{1}{2}$, $w_1 = \frac{1}{2}$, $w_2 = 0$, to the linear program:

$$\text{Maximize } x_1,$$

subject to:

$$\begin{aligned} x_1 + x_2 &\leq 3 && [a_{ij} \text{ constraint}], \\ \left.\begin{aligned} x_1 - x_2 &\leq 1 \\ x_2 &\leq 2 \end{aligned}\right\} && && [e_{ij} \text{ constraints}], \\ x_1 \geq 0, \quad & x_2 \geq 0. \end{aligned}$$

c) Show that an optimal solution $\overline{x}_j$ for $j = 1, 2, \ldots, n$ for the ''Lagrangian problem'' from part (a) need not be optimal for the original problem [see the example in part (b)]. Under what conditions is an optimal solution to the Lagrangian problem optimal in the original problem?

d) We know that if the ''Lagrangian problem'' has an optimal solution, then it has an extreme-point solution. Does this imply that there is an extreme point to the Lagrangian problem that is optimal in the original problem?

25. The payoff matrix (a_{ij}) for a two-person zero-sum game is said to be *skew symmetric* if the matrix has as many rows as columns and $a_{ij} = -a_{ji}$ for each choice of i and j. The payoff matrix for the game Rock, Paper and Scissors discussed in Exercise 20 has this property.

a) Show that if players 1 and 2 use the same strategy,

$$x_1 = y_1, \quad x_2 = y_2, \quad \ldots, \quad x_n = y_n,$$

then the payoff

$$\sum_{i=1}^{n} \sum_{j=1}^{n} y_i a_{ij} x_j$$

from the game is zero.

b) Given any strategy $x_1, x_2, \ldots, x_n$ for player 2, i.e., satisfying:

$$\begin{array}{rcccccccl} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & \geq & w, \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & \geq & w, \\ \vdots & & & & & & \vdots & & \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & \geq & w, \\ x_1 & + & x_2 & + & \cdots & + & x_n & = & 1, \end{array}$$

multiply the first n constraints by $y_1 = x_1$, $y_2 = x_2, \ldots,$ and $y_n = x_n$, respectively, and add. Use this manipulation, together with part (a), to show that $w \leq 0$. Similarly, show that the value to the player 1 problem satisfies $v \geq 0$. Use linear-programming duality to conclude that $v = w = 0$ is the value of the game.

26. In Section 4.9 we showed how to use linear-programming duality theory to model the column and row players' decision-making problems in a zero-sum two-person game. We would now like to exhibit a stronger connection between linear-programming duality and game theory.

Consider the linear-programming dual problems

$$\text{Maximize } z = \sum_{j=1}^{n} c_j x_j, \qquad \text{Minimize } w = \sum_{i=1}^{m} y_i b_i,$$

subject to: subject to

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i, \qquad\qquad \sum_{i=1}^{m} y_i a_{ij} \ge c_j,$$

$$x_j \ge 0, \qquad\qquad y_i \ge 0,$$

where the inequalities apply for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$; in addition, consider a zero-sum two-person game with the following payoff table:

$\overline{y}_1$	$\overline{y}_2$	$\cdots$	$\overline{y}_m$	$\overline{x}_1$	$\overline{x}_2$	$\cdots$	$\overline{x}_m$	t
0	0	$\cdots$	0	a_{11}	a_{12}	$\cdots$	a_{1n}	$-b_1$
0	0	$\cdots$	0	a_{21}	a_{22}	$\cdots$	a_{2n}	$-b_2$
$\vdots$				$\vdots$				$\vdots$
0	0	$\cdots$	0	a_{m1}	a_{m2}	$\cdots$	a_{mn}	$-b_m$
$-a_{11}$	$-a_{21}$	$\cdots$	$-a_{m1}$	0	0	$\cdots$	0	c_1
$-a_{12}$	$-a_{22}$	$\cdots$	$-a_{m2}$	0	0	$\cdots$	0	c_2
$\vdots$				$\vdots$			$\vdots$	
$-a_{1n}$	$-a_{2n}$	$\cdots$	$-a_{mn}$	0	0	$\cdots$	0	c_n
b_1	b_2	$\cdots$	b_m	$-c_1$	$-c_2$	$\cdots$	$-c_n$	0

The quantities $\overline{x}_j$, $\overline{y}_i$, and t shown above the table are the column players' selection probabilities.

a) Is this game skew symmetric? What is the value of the game? (Refer to the previous exercise.)

b) Suppose that x_j for $j = 1, 2, \ldots, n$ and y_i for $i = 1, 2, \ldots, m$ solve the linear-programming primal and dual problems. Let

$$t = \frac{1}{\sum_{j=1}^{n} x_j + \sum_{i=1}^{m} y_i + 1},$$

let $\overline{x}_j = tx_j$ for $j = 1, 2, \ldots, n$, and let $\overline{y}_i = ty_i$ for $i = 1, 2, \ldots, m$. Show that $\overline{x}_j$, $\overline{y}_i$, and t solve the given game.

c) Let $\overline{x}_j$ for $j = 1, 2, \ldots, n$, $\overline{y}_i$ for $i = 1, 2, \ldots, m$, and t solve the game, and suppose that $t > 0$. Show that

$$x_j = \frac{\overline{x}_j}{t} \quad \text{and } y_i = \frac{\overline{y}_i}{t}$$

solve the primal and dual linear programs. (See *Hint*.)

[*Hint*: Use the value of the game in parts (b) and (c), together with the conditions for strong duality in linear programming.]

27. Suppose we approach the solution of the usual linear program:

$$\text{Maximize } \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\sum_{j=1}^{n} c_{ij}x_j \le b_i \quad (i = 1, 2, \ldots, m), \tag{30}$$

$$x_j \ge 0 \quad (j = 1, 2, \ldots, n),$$

by the method of Lagrange multipliers. Define the Lagrangian maximization problem as follows:

$$L(\lambda_1, \lambda_2, \ldots, \lambda_m) = \text{Maximize } \left\{ \sum_{j=1}^{n} c_j x_j - \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} c_{ij}x_j - b_i \right) \right\}$$

subject to:

$$x_j \geq 0 \qquad (j = 1, 2, \ldots, n).$$

Show that the dual of (30) is given by:

$$\text{Minimize } L(\lambda_1, \lambda_2, \ldots, \lambda_m),$$

subject to:

$$\lambda_i \geq 0 \quad (i = 1, 2, \ldots, m).$$

[*Note*: The Lagrangian function $L(\lambda_1, \lambda_2, \ldots, \lambda_m)$ may equal $+\infty$ for certain choices of the Lagrange multipliers $\lambda_1, \lambda_2, \ldots, \lambda_m$. Would these values for the Lagrange multipliers ever be selected when minimizing $L(\lambda_1, \lambda_2, \ldots, \lambda_m)$?]

28. Consider the bounded-variable linear program:

$$\text{Maximize } \sum_{j=1}^{n} c_j x_j,$$

subject to:

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \quad (i = 1, 2, \ldots, m),$$

$$\ell_j \leq x_j \leq u_j \quad (j = 1, 2, \ldots, n).$$

Define the Lagrangian maximization problem by:

$$L(\lambda_1, \lambda_2, \ldots, \lambda_m) = \text{Maximize } \left\{ \sum_{j=1}^{n} c_j x_j - \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} a_{ij} x_j - b_i \right) \right\}$$

subject to:

$$\ell_j \leq x_j \leq u_j \qquad (j = 1, 2, \ldots, n).$$

Show that the primal problem is bounded above by $L(\lambda_1, \lambda_2, \ldots, \lambda_m)$ so long as $\lambda_i \geq 0$ for $i = 1, 2, \ldots, m$.

29. Consider the bounded-variable linear program and the corresponding Lagrangian maximization problem defined in Exercise 28.

a) Show that:

$$\text{Maximize } \sum_{j=1}^{n} c_j x_j \qquad = \qquad \text{Minimize} L(\lambda_1, \lambda_2, \ldots, \lambda_m),$$

subject to: subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \qquad \lambda_i \geq 0,$$

$$\ell_j \leq x_j \leq u_j, \qquad ,$$

which is a form of strong duality.

b) Why is it that $L(\lambda_1, \lambda_2, \ldots, \lambda_m)$ in this case does not reduce to the usual linear-programming dual problem?

30. Suppose that we define the function:

$$f(x_1, x_2, \ldots, x_n;\ y_1, y_2, \ldots, y_m) = \sum_{j=1}^{n} c_j x_j - \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_j y_j + \sum_{i=1}^{m} b_i y_i.$$

Show that the minimax property

$$\underset{x_j \geq 0}{\text{Max}} \ \underset{y_j \geq 0}{\text{Min}} \ f(x_1, \ldots, x_n;\ y_1, \ldots, y_m) = \underset{y_j \geq 0}{\text{Min}} \ \underset{x_j \geq 0}{\text{Max}} \ f(x_1, \ldots, x_n;\ y_1, \ldots, y_m)$$

implies the strong duality property.

ACKNOWLEDGMENTS

In Exercise 6, the pill interpretation of the diet problem appears in *The Theory of Linear Economic Models*, by David Gale, McGraw-Hill Book Company, Inc., 1960.
Exercises 8 and 11 are based on material contained in *Mathematical Programming and the Analysis of Capital Budgeting Problems*, by H. Martin Weingartner, Prentice-Hall, Inc. 1963.
Exercise 12 is based on the Gondon Enterprises case written by Paul W. Marshall and Colin C. Blaydon.