Chapter 4:
Linear Programming Sensitivity Analysis

What if there is uncertainly about one or more values in the LP model?

<u>Sensitivity analysis</u> allows us to determine how "sensitive" the optimal solution is to changes in data values.

This includes analyzing changes in:

1. An Objective Function Coefficient (OFC)
2. A Right Hand Side (RHS) value of a constraint

Graphical Sensitivity Analysis

We can use the graph of an LP to see what happens when:

1. An OFC changes, or
2. A RHS changes

Recall the Flair Furniture problem

Flair Furniture Problem

Max $7T + 5C$ (profit)

Subject to the constraints:

$3T + 4C \leq 2400$	(carpentry hrs)
$2T + 1C \leq 1000$	(painting hrs)
$C \leq 450$	(max # chairs)
$T \geq 100$	(min # tables)
$T, C \geq 0$	(nonnegativity)

Objective Function Coefficient (OFC) Changes

What if the profit contribution for tables changed from \$7 to \$8 per table?

8

Max ~~7~~ X T + 5 C (profit)

Clearly profit goes up, but would we want to make more tables and less chairs? (i.e. Does the optimal solution change?)

Characteristics of OFC Changes

- There is no effect on the feasible region
- The slope of the level profit line changes
- If the slope changes enough, a different corner point will become optimal

Original

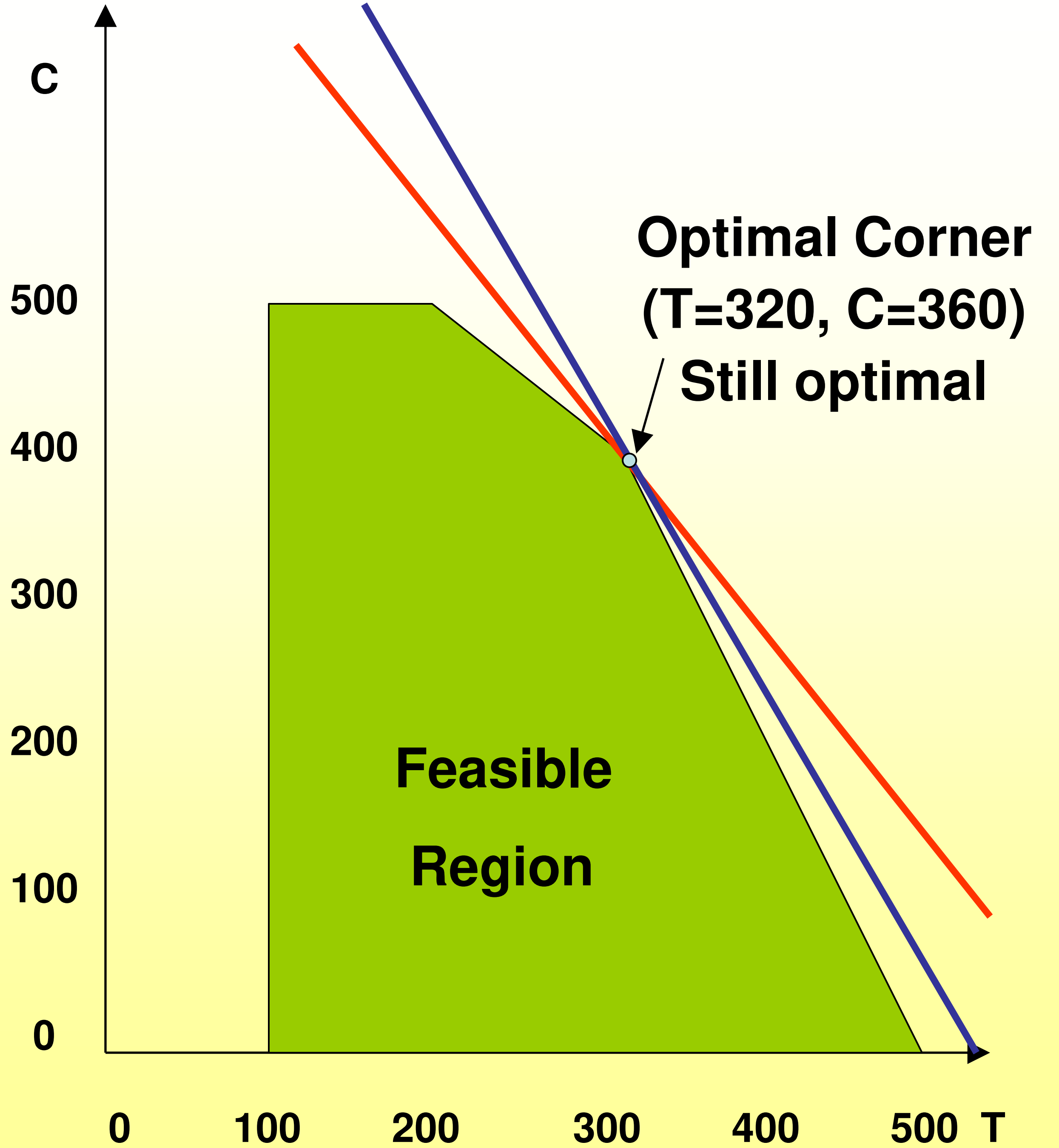
Objective Function

7T + 5 C = $4040

Revised

Objective Function

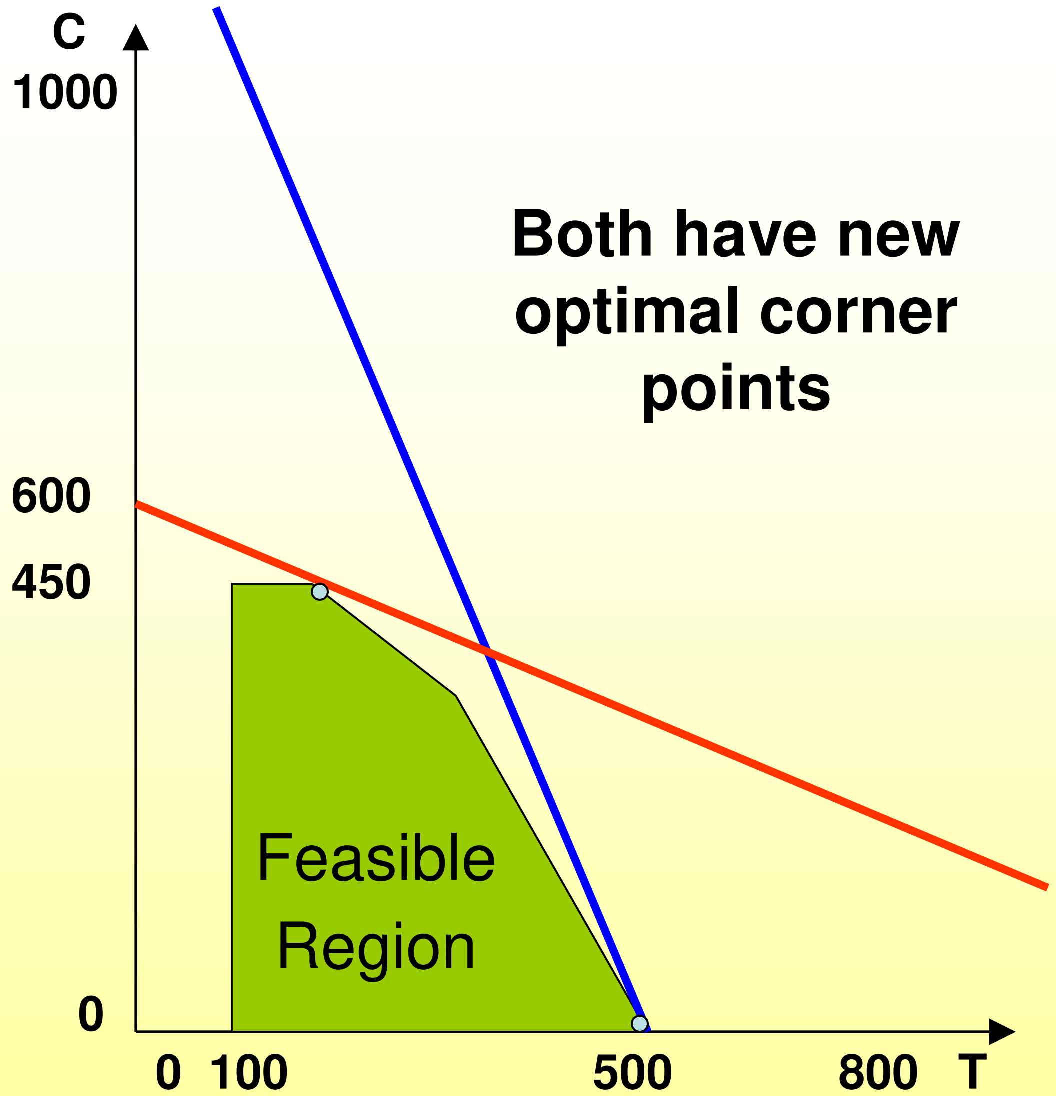
8T + 5 C = $4360



What if the OFC became higher? Or lower?

11T + 5C = $5500
<u>Optimal Solution</u>
(T=500, C=0)

3T + 5C = $2850
<u>Optimal Solution</u>
(T=200, C=450)



- There is a range for each OFC where the current optimal corner point remains optimal.
- If the OFC changes beyond that range a new corner point becomes optimal.
- Excel’s Solver will calculate the OFC range.

Right Hand Side (RHS) Changes

What if painting hours available changed from 1000 to 1300?

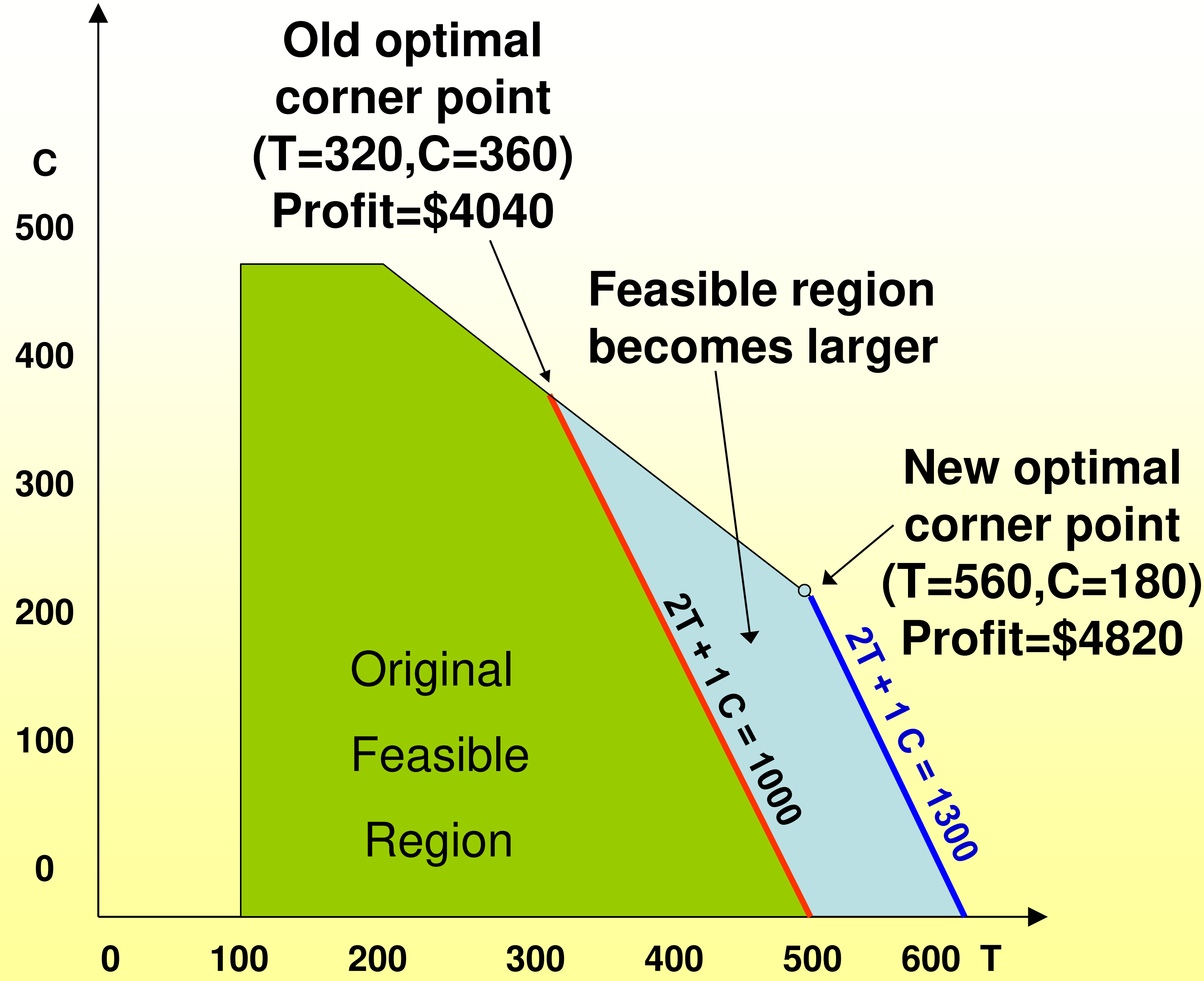
1300

$2T + 1C \leq 1000$ (painting hrs)

This increase in resources could allow us to increase production and profit.

Characteristics of RHS Changes

- The constraint line shifts, which could change the feasible region
- Slope of constraint line does not change
- Corner point locations can change
- The optimal solution can change



Old optimal
corner point
(T=320,C=360)
Profit=$4040
Feasible region
becomes larger
New optimal
corner point
(T=560,C=180)
Profit=$4820
C
500
400
300
200
100
0
Original
Feasible
Region
2T + 1 C = 1000
2T + 1 C = 1300
0
100
200
300
400
500
600
T

Effect on Objective Function Value

New profit	= $4,820
Old profit	= $4,040
Profit increase	= $780 from 300 additional painting hours

$2.60 in profit per hour of painting

- Each additional hour will increase profit by $2.60
- Each hour lost will decrease profit by $2.60

Shadow Price

The change is the objective function value per one-unit increase in the RHS of the constraint.

Will painting hours be worth $2.60 per hour regardless of many hours are available ?

Range of Shadow Price Validity

Beyond some RHS range the value of each painting hour will change.

While the RHS stays within this range, the shadow price does not change.

Excel will calculate this range as well as the shadow price.

Solver's Sensitivity Report

When Excel Solver is used to find an optimal solution, the option of generating the "Sensitivity Report" is available.

Go to file 4-1.xls

Constraint RHS Changes

If the change in the RHS value is within the allowable range, then:

- The shadow price does not change
- The change in objective function value =

  (shadow price) x (RHS change)

If the RHS change goes beyond the allowable range, then the shadow price will change.

Objective Function Coefficient (OFC) Changes

If the change in OFC is within the allowable range, then:

- The optimal solution does not change
- The new objective function value can be calculated

Anderson Electronics Example

Decision: How many of each of 4 products to make?

Objective: Maximize profit

Decision Variables:

V = number of VCR's
S = number of stereos
T = number of TV's
D = number of DVD players

Max $29V + 32S + 72T + 54D$ (in $ of profit)

Subject to the constraints:

$$3V + 4S + 4T + 3D \leq 4700 \quad \text{(elec. components)}$$

$$2V + 2S + 4T + 3D \leq 4500 \quad \text{(nonelec. components)}$$

$$V + S + 3T + 2D < 2500 \quad \text{(assembly hours)}$$

$$V, S, T, D \geq 0 \quad \text{(nonnegativity)}$$

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RHS Change Questions

- What if the supply of nonelectrical components changes?

- What happens if the supply of electrical components
  - increased by 400 (to 5100)?
  - increased by 4000 (to 8700)?

- What if we could buy an additional 400 elec. components for $1 more than usual? Would we want to buy them?

- What if would could get an additional 250 hours of assembly time by paying $5 per hour more than usual? Would this be profitable?

Decision Variables That Equal 0

We are not currently making any VCR’s ($V=0$) because they are not profitable enough.

How much would profit need to increase before we would want to begin making VCR’s?

Reduced Cost
of a Decision Variable

(marginal contribution to the obj. func. value)

- (marginal value of resources used)

= **Reduced Cost**

marginal profit of a VCR	=	$29
- marginal value of resources	=	?
Reduced Cost of a VCR	=	- $1.0

Reduced Cost is:

- The minimum amount by which the OFC of a variable should change to cause that variable to become non-zero.

- The amount by which the objective function value would change if the variable were forced to change from 0 to 1.

OFC Change Questions

- For what range of profit contributions for DVD players will the current solution remain optimal?
- What happens to profit if this value drops to $50 per DVD player?

Alternate Optimal Solutions

May be present when there are 0’s in the Allowable Increase or Allowable Decrease values for OFC values.

Simultaneous Changes

All changes discussed up to this point have involved only 1 change at a time.

What if several OFC's change?

Or

What if several RHS's change?

Note: they cannot be mixed

The 100% Rule

$$\sum \text{(change / allowable change)} \leq 1$$

RHS Example

- Electrical components decrease 500

  500 / 950 = 0.5263

- Assembly hours increase 200

  200 / 466.67 = 0.4285

  0.9548

The sensitivity report can still be used

Pricing New Variables

Suppose they are considering selling a new product, Home Theater Systems (HTS)

Need to determine whether making HTS's would be sufficiently profitable

Producing HTS's would take limited resources away from other products

- To produce one HTS requires:

  5 electrical components

  4 nonelectrical components

  4 hours of assembly time

- Can shadow prices be used to calculate reduction in profit from other products? (check 100% rule)

  $5/950 + 4/560 + 4/1325 = 0.015 < 1$

Required Profit Contribution per HTS

elec cpnts	5 x $ 2	=	$10
nonelec cpnts	4 x $ 0	=	$ 0
assembly hrs	4 x $24	=	$96
			$106

Shadow Prices

Making 1 HTS will reduce profit (from other products) by $106

- Need (HTS profit contribution) $\geq$ \$106
- Cost to produce each HTS:

elec cpnts	5 x \$ 7 =	\$35
nonelec cpnts	4 x \$ 5 =	\$20
assembly hrs	4 x \$10 =	\$40
		\$95

(HTS profit contribution) = (selling price) - \$95

So selling price must be at least \$201

Is HTS Sufficiently Profitable?

- Marketing estimates that selling price should not exceed $175
- Producing one HTS will cause profit to fall by $26 ($201 - $175)

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Sensitivity Analysis for a Minimization Problem

Burn-Off makes a “miracle” diet drink

Decision: How much of each of 4 ingredients to use?

Objective: Minimize cost of ingredients

Data

Units of Chemical per Ounce of Ingredient					
Chemical	Ingredient				Requirement
	A	B	C	D	
X	3	4	8	10	≥ 280 units
Y	5	3	6	6	≥ 200 units
Z	10	25	20	40	≤ 1050 units

$ per ounce of ingredient			
$0.40	$0.20	$0.60	$0.30

$$\text{Min } 0.40A + 0.20B + 0.60C + 0.30D \quad (\$ \text{ of cost})$$

Subject to the constraints

$$A + B + C + D \geq 36 \quad \text{(min daily ounces)}$$

$$3A + 4B + 8C + 10D \geq 280 \quad \text{(chem x min)}$$

$$5A + 3B + 6C + 6D \geq 200 \quad \text{(chem y min)}$$

$$10A + 25B + 20C + 40D \leq 280 \quad \text{(chem z max)}$$

$$A, B, C, \geq 0$$

Go to file 4-5.xls