Chapter 4:
Linear Programming
Sensitivity Analysis
What if there is uncertainly about one or more values in the LP model?

Sensitivity analysis allows us to determine how “sensitive” the optimal solution is to changes in data values.

This includes analyzing changes in:
1. An Objective Function Coefficient (OFC)
2. A Right Hand Side (RHS) value of a constraint
Graphical Sensitivity Analysis

We can use the graph of an LP to see what happens when:

1. An OFC changes, or
2. A RHS changes

Recall the Flair Furniture problem
Flair Furniture Problem

Max $7T + 5C$ (profit)

Subject to the constraints:

$$3T + 4C \leq 2400$$ (carpentry hrs)

$$2T + 1C \leq 1000$$ (painting hrs)

$$C \leq 450$$ (max # chairs)

$$T \geq 100$$ (min # tables)

$$T, C \geq 0$$ (nonnegativity)
Objective Function Coefficient (OFC) Changes

What if the profit contribution for tables changed from $7 to $8 per table?

Max \( \times T + 5 C \) (profit)

Clearly profit goes up, but would we want to make more tables and less chairs? (i.e. Does the optimal solution change?)
Characteristics of OFC Changes

- There is no effect on the feasible region
- The slope of the level profit line changes
- If the slope changes enough, a different corner point will become optimal
Original Objective Function
$7T + 5C = 4040$

Revised Objective Function
$8T + 5C = 4360$

Optimal Corner
(T=320, C=360)
Still optimal
What if the OFC became higher? Or lower?

\[11T + 5C = \$5500\]
**Optimal Solution**
\(T=500, \ C=0\)

\[3T + 5C = \$2850\]
**Optimal Solution**
\(T=200, \ C=450\)

Both have new optimal corner points
• There is a range for each OFC where the current optimal corner point remains optimal.

• If the OFC changes beyond that range a new corner point becomes optimal.

• Excel’s Solver will calculate the OFC range.
Right Hand Side (RHS) Changes

What if painting hours available changed from 1000 to 1300?

\[ 2T + 1C \leq 1000 \]

This increase in resources could allow us to increase production and profit.
Characteristics of RHS Changes

• The constraint line shifts, which could change the feasible region

• Slope of constraint line does not change

• Corner point locations can change

• The optimal solution can change
Feasible region becomes larger

Old optimal corner point
(T=320, C=360)
Profit=$4040

New optimal corner point
(T=560, C=180)
Profit=$4820

Old optimal corner point
(T=320, C=360)
Profit=$4040

Feasible region becomes larger

New optimal corner point
(T=560, C=180)
Profit=$4820
Effect on Objective Function Value

New profit = $4,820
Old profit = $4,040
Profit increase = $780 from 300 additional painting hours

$2.60 in profit per hour of painting

• Each additional hour will increase profit by $2.60
• Each hour lost will decrease profit by $2.60
Shadow Price

The change is the objective function value per one-unit increase in the RHS of the constraint.

Will painting hours be worth $2.60 per hour regardless of many hours are available?
Range of Shadow Price Validity

Beyond some RHS range the value of each painting hour will change.

While the RHS stays within this range, the shadow price does not change.

Excel will calculate this range as well as the shadow price.
Solver’s Sensitivity Report

When Excel Solver is used to find an optimal solution, the option of generating the “Sensitivity Report” is available.

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Constraint RHS Changes

If the change in the RHS value is within the allowable range, then:

• The shadow price does not change
• The change in objective function value = (shadow price) x (RHS change)

If the RHS change goes beyond the allowable range, then the shadow price will change.
Objective Function Coefficient (OFC) Changes

If the change in OFC is within the allowable range, then:

• The optimal solution does not change
• The new objective function value can be calculated
Anderson Electronics Example

Decision: How many of each of 4 products to make?

Objective: Maximize profit

Decision Variables:
- \( V \) = number of VCR’s
- \( S \) = number of stereos
- \( T \) = number of TV’s
- \( D \) = number of DVD players
Max $29V + 32S + 72T + 54D$ (in $\$ $ of profit)

Subject to the constraints:

\begin{align*}
3V + 4S + 4T + 3D & \leq 4700 \quad \text{(elec. components)} \\
2V + 2S + 4T + 3D & \leq 4500 \quad \text{(nonelec. components)} \\
V + S + 3T + 2D & < 2500 \quad \text{(assembly hours)} \\
V, S, T, D & \geq 0 \quad \text{(nonnegativity)}
\end{align*}

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RHS Change Questions

• What if the supply of nonelectrical components changes?

• What happens if the supply of electrical components
  – increased by 400 (to 5100)?
  – increased by 4000 (to 8700)?
• What if we could buy an additional 400 elec. components for $1 more than usual? Would we want to buy them?

• What if we could get an additional 250 hours of assembly time by paying $5 per hour more than usual? Would this be profitable?
Decision Variables That Equal 0

We are not currently making any VCR’s (V=0) because they are not profitable enough.

How much would profit need to increase before we would want to begin making VCR’s?
Reduced Cost of a Decision Variable

(marginal contribution to the obj. func. value)
- (marginal value of resources used)

= Reduced Cost

marginal profit of a VCR = $29
- marginal value of resources = ?

Reduced Cost of a VCR = - $1.0
Reduced Cost is:

• The minimum amount by which the OFC of a variable should change to cause that variable to become non-zero.

• The amount by which the objective function value would change if the variable were forced to change from 0 to 1.
OFC Change Questions

• For what range of profit contributions for DVD players will the current solution remain optimal?

• What happens to profit if this value drops to $50 per DVD player?
Alternate Optimal Solutions

May be present when there are 0’s in the Allowable Increase or Allowable Decrease values for OFC values.
Simultaneous Changes

All changes discussed up to this point have involved only 1 change at a time.

What if several OFC’s change?

Or

What if several RHS’s change?

Note: they cannot be mixed
The 100% Rule

\[ \sum \left( \frac{\text{change}}{\text{allowable change}} \right) \leq 1 \]

RHS Example

- Electrical components decrease 500
  \[ \frac{500}{950} = 0.5263 \]

- Assembly hours increase 200
  \[ \frac{200}{466.67} = 0.4285 \]

The sensitivity report can still be used
Pricing New Variables

Suppose they are considering selling a new product, Home Theater Systems (HTS)

Need to determine whether making HTS’s would be sufficiently profitable

Producing HTS’s would take limited resources away from other products
• To produce one HTS requires:
  5 electrical components
  4 nonelectrical components
  4 hours of assembly time

• Can shadow prices be used to calculate reduction in profit from other products?
  \[
  \frac{5}{950} + \frac{4}{560} + \frac{4}{1325} = 0.015 < 1
  \]
Required Profit Contribution per HTS

- **elec cpnts**: $10
  \[ 5 \times \$2 = \$10 \]
- **nonelec cpnts**: $0
  \[ 4 \times \$0 = \$0 \]
- **assembly hrs**: $96
  \[ 4 \times \$24 = \$96 \]

Making 1 HTS will reduce profit (from other products) by $106.
• Need (HTS profit contribution) ≥ $106
• Cost to produce each HTS:
  elec cpnts  5 x $7 = $35
  nonelec cpnts  4 x $5 = $20
  assembly hrs  4 x $10 = $40

HTS profit contribution = (selling price) - $95

So selling price must be at least $201
Is HTS Sufficiently Profitable?

• Marketing estimates that selling price should not exceed $175

• Producing one HTS will cause profit to fall by $26 ($201 - $175)

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Sensitivity Analysis for a Minimization Problem

Burn-Off makes a “miracle” diet drink

**Decision:** How much of each of 4 ingredients to use?

**Objective:** Minimize cost of ingredients
### Units of Chemical per Ounce of Ingredient

<table>
<thead>
<tr>
<th>Chemical</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>Requirement</th>
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<tbody>
<tr>
<td>X</td>
<td>3</td>
<td>4</td>
<td>8</td>
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<tr>
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<td>5</td>
<td>3</td>
<td>6</td>
<td>6</td>
<td>$\geq 200$ units</td>
</tr>
<tr>
<td>Z</td>
<td>10</td>
<td>25</td>
<td>20</td>
<td>40</td>
<td>$\leq 1050$ units</td>
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</tbody>
</table>

#### $ per ounce of ingredient 

<table>
<thead>
<tr>
<th></th>
<th>$0.40$</th>
<th>$0.20$</th>
<th>$0.60$</th>
<th>$0.30$</th>
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</thead>
<tbody>
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</tbody>
</table>
Min $0.40A + 0.20B + 0.60C + 0.30D$ ($ of cost$)

Subject to the constraints

$A + B + C + D \geq 36$ (min daily ounces)

$3A + 4B + 8C + 10D \geq 280$ (chem x min)

$5A + 3B + 6C + 6D \geq 200$ (chem y min)

$10A + 25B + 20C + 40D \leq 280$ (chem z max)

$A, B, C, \geq 0$

Go to file 4-5.xls