
The Two-Phase Simplex Method

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Preview

- When a basic feasible solution is not readily available, the two-phase simplex method may be used as an alternative to the big M method.
 - In the two-phase simplex method, we add artificial variables to the same constraints as we did in big M method. Then we find a bfs to the original LP by solving the Phase I LP.
 - In the Phase I LP, the objective function is to minimize the sum of all artificial variables.
 - At the completion of Phase I, we reintroduce the original LP's objective function and determine the optimal solution to the original LP.
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The two-phase simplex method

- Step 1

Modify the constraints so that the right-hand side of each constraint is nonnegative. This requires that each constraint with a negative right-hand side be multiplied through by -1.

- Step 1'

Identify each constraint that is now an = or \geq constraint. In step 3, we will add an the artificial variable to each of these constraints.

- Step 2

Convert each inequality constraint to standard form.

For \leq constraint i , we add a slack variable s_i ;

For \geq constraint i , we add an excess variable e_i ;

The two-phase simplex method

- Step 4

For now, ignore the original LP's objective function. Instead solve an LP whose objective function is $\min w' = (\text{sum of all the artificial variables})$. This is called the **Phase I LP**. The act of solving the phase I LP will force the artificial variables to be zero.

The two-phase simplex method

Since each $a_i \geq 0$, solving the Phase I LP will result in one of the following three cases:

- Case 1

The optimal value of w' is greater than zero. In this case, the original LP has no feasible solution.

- Case 2

The optimal value of w' is equal to zero, and no artificial variables are in the optimal Phase I basis. In this case, we drop all columns in the optimal Phase I tableau that correspond to the artificial variables. We now combine the original objective function with the constraints from the optimal Phase I tableau. This yields the Phase II LP. The optimal solution to the Phase II LP is the optimal solution to the original LP.

The two-phase simplex method

■ Case 3

The optimal value of w' is equal to zero and at least one artificial variable is in the optimal Phase I basis. In this case, we can find the optimal solution to the original LP if at the end of Phase I we drop from the optimal Phase I tableau all nonbasic artificial variables and any variable from the original problem that has a negative coefficient in row 0 of the optimal Phase I tableau.

Phase I and II feasible solutions

- Suppose the original LP is infeasible. Then the only way to obtain a feasible solution to the Phase I LP is to let at least one artificial variable be positive. In this situation, $w' > 0$ will result.
 - On the other hand, if the original LP has a feasible solution, this feasible solution is feasible in the Phase I LP and yields $w' = 0$. This means that if the original LP has a feasible solution, the optimal Phase I solution will have $w' = 0$.
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Example 5

- The Bevco problem

$$\min z = 2x_1 + 3x_2$$

$$\text{s.t. } \frac{1}{2}x_1 + \frac{1}{4}x_2 \leq 4 \quad (\text{sugar constraint})$$

$$x_1 + 3x_2 \geq 20 \quad (\text{vitamin C constraint})$$

$$x_1 + x_2 = 10 \quad (\text{10 oz in bottle of Oranj})$$

$$x_1, x_2 \geq 0$$

Example 5

■ Solution

As in the Big M method, step 1-3 transform the constraints into

$$\begin{array}{rcl} 1/2 x_1 + 1/4 x_2 + s_1 & & = 4 \\ x_1 + 3x_2 - e_2 + a_2 & & = 20 \\ x_1 + x_2 & & + a_3 = 10 \end{array}$$

Example 5

Step 4 yields the following Phase I LP:

$$\begin{array}{rcll} \min w' = & & a_2 + a_3 & \\ \text{s.t. } & 1/2 x_1 + 1/4 x_2 + s_1 & & = 4 \\ & x_1 + 3x_2 - e_2 + a_2 & & = 20 \\ & x_1 + x_2 & + a_3 & = 10 \end{array}$$

This set of equations yields a starting bfs for Phase I ($s_1=4, a_2=20, a_3=10$).

Example 5

■ Note:

The row 0 for this tableau ($w' - a_2 - a_3 = 0$) contains the basic variables a_2 and a_3 . as in the Big M method, a_2 and a_3 must be eliminated from row 0 before we can solve Phase I.

To eliminate a_2 and a_3 from row 0,

$$\begin{array}{rcll} \text{Row 0:} & w' & & -a_2 - a_3 = 0 \\ +\text{Row 2:} & & x_1 + 3x_2 - e_2 + a_2 & = 20 \\ +\text{Row 3:} & & x_1 + x_2 & + a_3 = 10 \\ =\text{New row 0:} & w' & +2x_1 + 4x_2 - e_2 & = 30 \end{array}$$

Example 5

Combining the new row 0 with the Phase I constraints yields the initial Phase I tableau:

Initial Phase I tableau for Bevco

W'	x_1	x_2	s_1	e_2	a_2	a_3	rhs	Basic variable	ratio
1	2	4	0	-1	0	0	30	$W'=30$	
0	$\frac{1}{2}$	$\frac{1}{4}$	1	0	0	0	4	$s_1=4$	16
0	1	③	0	-1	1	0	20	$a_2=20$	$20/3^*$
0	1	1	0	0	0	1	10	$a_3=10$	10

Note: Phase I problem is always a min problem, even if the original LP is a max problem.

Example 5

After performing the necessary ero's, we obtain the second tableau.

Phase I tableau for Bevco after one iteration

W'	x₁	x₂	s₁	e₂	a₂	a₃	rhs	Basic variable	ratio
1	2/3	0	0	1/3	-4/3	0	10/3	W'=10/3	
0	5/12	0	1	1/12	-1/12	0	7/3	s ₁ =7/3	28/5
0	1/3	1	0	-1/3	1/3	0	20/3	x ₂ =20/3	20
0	2/3	0	0	1/3	-1/3	1	10/3	a ₃ =10/3	5*

Example 5

After performing the necessary ero's, we obtain the third tableau.

Phase I tableau for Bevco after two iteration

W'	x₁	x₂	s₁	e₂	a₂	a₃	rhs	Basic variable
1	0	0	0	0	-1	-1	0	W'=0
0	0	0	1	-1/8	1/8	-5/8	1/4	s ₁ =1/4
0	0	1	0	-1/2	1/2	-1/2	5	x ₂ =5
0	1	0	0	1/2	-1/2	3/2	5	x ₁ =5

Example 5

Since $w'=0$, Phase I has been concluded. The basic feasible solution $s_1=1/4, x_2=5, x_1=5$ has been found.

Since no artificial variables are in the optimal Phase I basis, the problem is an example of Case 2.

We now drop the columns for the artificial variables a_2 and a_3 and reintroduce the original objective function.

$$\min z=2x_1+3x_2 \text{ or } z-2x_1-3x_2=0$$

Example 5

Since x_1 and x_2 are both in the optimal Phase I basis, they must be eliminated from the Phase II row 0.

$$\begin{aligned} \text{Phase II row 0:} & \quad z - 2x_1 - 3x_2 & = 0 \\ +3(\text{row 2}): & & 3x_2 - 3/2e_2 = 15 \\ +2(\text{row 3}): & \quad 2x_1 & + e_2 = 10 \\ = \text{New Phase II row 0:} & \quad z & - 1/2e_2 = 25 \end{aligned}$$

We now begin Phase II with the following set of equations:

$$\begin{aligned} \min z & \quad -1/2e_2 = 25 \\ \text{s.t} & \quad s_1 - 1/8e_2 = 1/4 \\ & \quad x_2 - 1/2e_2 = 5 \\ & \quad x_1 + 1/2e_2 = 5 \end{aligned}$$

This is optimal.

The optimal solution is $z=25$ $x_1=5$ $x_2=5$ $s_1=1/4$ $e_2=0$.

Example 6

- We modify Bevco's problem that 36 mg of vitamin C are required. Then, we know that this problem is infeasible. We begin with the original problem:

$$\min z = 2x_1 + 3x_2$$

$$\text{s.t. } \frac{1}{2}x_1 + \frac{1}{4}x_2 \leq 4 \quad (\text{sugar constraint})$$

$$x_1 + 3x_2 \geq 36 \quad (\text{vitamin C constraint})$$

$$x_1 + x_2 = 10 \quad (10 \text{ oz in bottle of Oranj})$$

$$x_1, x_2 \geq 0$$

Example 6

■ Solution

After completing steps 1-4 of the two-phase simplex, we obtain the following Phase I problem:

$$\begin{aligned} \min w' &= a_2 + a_3 \\ \text{s.t. } & \frac{1}{2}x_1 + \frac{1}{4}x_2 + s_1 = 4 \\ & x_1 + 3x_2 - e_2 + a_2 = 36 \\ & x_1 + x_2 + a_3 = 10 \end{aligned}$$

From this set of equations, we see that the initial Phase I bfs is $s_1=4, a_2=36$ and $a_3=10$.

Example 6

Since the basic variables a_2 and a_3 occur in the Phase I objective function, they must be eliminated from the Phase I row 0.

$$\begin{array}{rcl} \text{Row 0:} & w' & -a_2 - a_3 = 0 \\ +\text{Row 2:} & x_1 + 3x_2 - e_2 + a_2 & = 36 \\ +\text{Row 3:} & x_1 + x_2 & + a_3 = 10 \\ =\text{New row 0:} & w' + 2x_1 + 4x_2 - e_2 & = 46 \end{array}$$

Example 6

With the new row 0, the initial Phase I tableau is:

W'	x_1	x_2	s_1	e_2	a_2	a_3	rhs	Basic variable	ratio
1	2	4	0	-1	0	0	46	$W'=46$	
0	$\frac{1}{2}$	$\frac{1}{4}$	1	0	0	0	4	$s_1=4$	16
0	1	3	0	-1	1	0	36	$a_2=36$	12
0	1	①	0	0	0	1	10	$a_3=10$	10^*

x_2 to enter, a_3 to leave the basis, the new tableau is:

W'	x_1	x_2	s_1	e_2	a_2	a_3	rhs	Basic variable
1	-2	0	0	-1	0	-4	6	$W'=6$
0	$\frac{1}{4}$	0	1	0	0	$-\frac{1}{4}$	$\frac{3}{2}$	$s_1=3/2$
0	-2	0	0	-1	1	-3	6	$a_2=6$
0	1	1	0	0	0	1	10	$x_2=10$

Example 6

- Since no variable in row 0 has a positive coefficient, this is an optimal Phase I tableau.
 - Since the optimal value of w' is $6 > 0$, the original LP must have no feasible solution.
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Remarks

- As with the Big M method, the column for any artificial variable may be dropped from future tableaus as soon as the artificial variable leaves the basis.
 - The Big M method and Phase I of the two-phase method make the same sequence of pivots. The two-phase method does not cause roundoff errors and other computational difficulties.
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